

The University of Chicago

SPACES ASSOCIATED
WITH NON-MODULAR MATRICES
WITH APPLICATION TO
RECIPROCAL

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1931

By

YUE KEI WONG

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CHICAGO, ILLINOIS

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INTRODUCTION

Since Carleman published his 'Sur Les Équations Intégrales Singulières à Noyau Réel et Symétrique', the interest in the study of non-limited matrices has been aroused. Naturally, we should expect to develop a certain theory for non-modular matrices in E. H. Moore's General Analysis. The present paper is a first attempt along this line. The object is to study the spaces associated with a T-matrix ¹, and to apply the results to the determination of reciprocals for non-modular and modular matrices on general ranges.

In the first chapter, a generalized Hellinger-Toeplitz theorem is established. The generalization from Hilbert space to modular space requires certain modifications which are not obtained without some difficulty. However, the difficulties are solved by the aid of 'modified functions'. After the theorem is established, we deduce an interesting theorem concerning the modularity of a given positive, hermitian matrix ϵ_0 relative to ϵ and the relation between their modular spaces. Several other theorems are given in this chapter.

Chapter II is devoted to the study of the properties of T-matrices and their associated spaces. In section IV A of his General Analysis, E. H. Moore develops a theory concerning matrices whose columns consist of modular functions. There the properties of the modular spaces, $\mathcal{M}^{\mu}$ and $\mathcal{M}^{2\mu}$, are studied. If the given matrix K^{12} on $\mathcal{P}'\mathcal{P}^2$ to $\mathcal{N}$ is modular relative to $\epsilon^1 \cdot \epsilon^2$,

¹ A T-matrix is one such that its columns and the conjugates of its rows consist of modular functions.

then the space $\mathcal{M}'^{k*}$ is contained in $\mathcal{M}'^k$; in other words, every Fourier coefficient function of μ^2 relative to K^{*21} is a modular function relative to ϵ_k^1 . If K^{12} is only T^{12} , such Fourier coefficient functions are not necessarily modular, and the interrelationship between the spaces, $\mathcal{M}'^k$ and $\mathcal{M}'^{k*}$, becomes far more complicated. For the present work, we are interested in the subspace of $\mathcal{M}'$ which is transformable by $J^2 K^{12}$ into $\mathcal{M}'$. Such a subspace is designated by $\mathcal{M}'^{2k'}$. Here a method is established for finding the space $\mathcal{M}'^{2k'}$ and other related spaces. Properties of these spaces are studied. Results concerning the interchangeability of two integration processes are secured. The theory developed in connection with this method enables us to study not only the problem of reciprocals, but also other topics.

In chapter III we study non-modular reciprocals for non-modular matrices. The definition given here is less restricted than that given by Moore, and indeed, his theory is a special case of this one.

Chapter IV is devoted to a special case. There we find the discussion of modular reciprocals for non-modular matrices. Finally, in chapter V, we investigate the inverse question of the preceding chapter, i. e., the case when modular matrices have non-modular reciprocals. More results are secured for these two special cases.

I. THE GENERALIZED HELLINGER-TOEPLITZ THEOREM

In this section, we shall generalize an interesting theorem given and demonstrated by Hellinger and Toeplitz in Hilbert space.¹ From this theorem we deduce some results which will have repeated application in our future development. The basis for the present section will be

$$\alpha^0, \varphi^1, \varphi^2, \epsilon^1, \epsilon^2,$$

where ϵ^1 and ϵ^2 are hermitian matrices of positive type on the ranges φ^1 and φ^2 respectively.

§ 1.

$$T(11)^2 K^{12} \overline{\text{rows}} M^2 \cdot J^2 K^{12} \mu^2 M^1 (\mu^2) \cdot \bar{B}(M^1 J^2 K^{12} \mu^2 | \mu^2 M^2 \mu^2 \leq 1) < \infty$$

We shall give an indirect proof following essentially the reasoning of Hellinger and Toeplitz with a few modifications.

If we put $\mu^2 = \epsilon^2(\cdot p^2)$, we secure from the second condition of the hypothesis that

$$J^2 K^{12} \epsilon^2(\cdot p^2) = K^{12}(\cdot p^2) M^1 (p^2)$$

Hence $K^{12} T^{12}$. The rest of the proof will be divided into six lemmas.³

$$1^0 \quad e \cdot \cdot \quad \exists (v_e^{2+}, \sigma_e') \ni M_{\sigma_e'}^1 J^2 K^{12} v_e^2 > e$$

¹ 'Grundlagen für eine Theorie der unendlichen Matrizen', Mathematischen Annalen, vol. 69 (1910), §10.

² Here and elsewhere we obtain a second theorem by interchanging the role of φ^1, φ^2 and ϵ^1, ϵ^2 . Usually the theorem by parity will not be written.

³ In the proof we use the notation μ^{2+} to indicate that $M^2 \mu^2 \leq 1$.

For if the conclusion is false, then

$$e \cdot). \quad \exists v_e^{2^{h+1}} \ni M^1 J^2 K^{1^2} v_e^2 > e.$$

By II A, T III, $\bar{B} M_{\sigma'}^1 J^2 K^{1^2} v_e^2 > e$. Therefore, $\exists \sigma'_e \ni M_{\sigma'_e}^1 J^2 K^{1^2} v_e^2 > e$.

For the rest of the proof, let us fix an arbitrary σ_0^2 .

$$ii^0 \quad \exists (v_h^{2^{h+1}} \cdot \sigma_h' \cdot \sigma_h^2 \mid h = 1, 2, \dots) \ni : h \geq 1 \quad).$$

$$(1) \quad M_{\sigma_h'}^1 J^2 K^{1^2} v_h^2 > 1 + 2^{h+1} (h + 1 + 2M_{\sigma_{h-1}}^{1^2} K_{\sigma_{h-1}}^{1^2}) \cdot$$

$$(2) \quad M^{1^2} (K_{\sigma_h'}^{1^2} - K_{\sigma_h' \sigma_h^2}^{1^2}) \leq 1 \cdot$$

$$(3) \quad \sigma_h^2 \supset \sigma_{h-1}^2.$$

To prove this lemma we show how to determine $(v_h^{2^{h+1}} \cdot \sigma_h' \cdot \sigma_h^2)$ with properties (1), (2), and (3) from σ_{h-1}^2 . Since we have σ_0^2 as a start, we secure a complete system having the properties (1), (2), and (3) for every h .

In fact, by i^0

$$\exists (v_h^{2^{h+1}} \cdot \sigma_h') \ni M_{\sigma_h'}^1 J^2 K^{1^2} v_h^2 > 1 + 2^{h+1} (h + 1 + 2M_{\sigma_{h-1}}^{1^2} K_{\sigma_{h-1}}^{1^2}).$$

With σ_h' determined, we secure by III A, T V and III B, T I

$$\exists \sigma_h^2 \supset \sigma_{h-1}^2 \ni M^{1^2} (K_{\sigma_h'}^{1^2} - K_{\sigma_h' \sigma_h^2}^{1^2}) \leq 1.$$

$$iii^0 \quad h \geq 1 \cdot). \quad M_{\sigma_h'}^1 J^2 K^{1^2} v_{h \sigma_h^2}^2 > 2^{h+1} (h + 1 + 2M_{\sigma_{h-1}}^{1^2} K_{\sigma_{h-1}}^{1^2}).$$

To prove this statement, let us write $v_{h \sigma_h^2}^2 = v_h^2 + v_{h \sigma_h^2}^2 - v_h^2$. Then by II A, T I, we have

$$a) \quad M_{\sigma_h'}^1 J^2 K^{1^2} v_{h \sigma_h^2}^2 \geq M_{\sigma_h'}^1 J^2 K^{1^2} v_h^2 - M_{\sigma_h'}^1 J^2 K^{1^2} (v_h^2 - v_{h \sigma_h^2}^2).$$

By T IV, and T VI in II A, and T V, and T II in III A, we obtain

$$b) \quad M_{\sigma_h'}^1 J^2 K^{1^2} (v_h^2 - v_{h \sigma_h^2}^2) = M_{\sigma_h'}^1 J^2 (K^{1^2} - K_{\sigma_h^2}^{1^2}) v_h^2 =$$

$$= M^1 J^E (K^{1E} - K_{\sigma_h^E}^{1E})_{\sigma_h^E} v_h^E \leq M^1 E (K_{\sigma_h^E}^{1E} - K_{\sigma_{h-1}^E}^{1E}) M^E v_h^E \leq 1.$$

Combining the inequalities a), b) with 1i⁰, we secure 1ii⁰.

$$1v^0 \quad \mu_0^E \equiv 0^E \cdot \left(\mu_h^E \equiv \mu_{h-1}^E + \frac{v_h^E \sigma_h^E - v_h^E \sigma_{h-1}^E}{2^{h+1}} \mid h = 1, 2, \dots \right) \cdot$$

$$(1) \quad \mu_g^E = \sum_h^{ig} (v_h^E \sigma_h^E - v_h^E \sigma_{h-1}^E) / 2^{h+1} \quad (g \geq 1),$$

$$(2) \quad \mu_g^E \sigma_g^E = \mu_g^E \quad (g \geq 1),$$

$$(3) \quad \mu_g^E (\sigma_f^E) = \mu_f^E (\sigma_f^E) \quad (g \geq f \geq 1),$$

$$(4) \quad \mu_g^E \uparrow^+ \quad (g \geq 1),$$

$$(5) \quad \exists \mu_{\star}^E \uparrow^+ \ni \mu_{\star}^E = \varprojlim_{g \geq 1} \mu_g^E$$

$$(6) \quad \mu_{\star}^E (\sigma_f^E) = \mu_f^E (\sigma_f^E) \quad (f \geq 1).$$

In fact, property (1) follows directly from the definition of μ_g^E , for

$$\sum_h^{ig} (v_h^E \sigma_h^E - v_h^E \sigma_{h-1}^E) 2^{-h-1} = \sum_h^{ig} (\mu_h^E - \mu_{h-1}^E) = \mu_g^E - \mu_0^E = \mu_g^E.$$

To prove (2), we make use of property (3) in 1i⁰, and II A, T VI, and obtain

$$(v_h^E \sigma_h^E)_{\sigma_g^E} = v_h^E \sigma_h^E \quad (g \geq h), \quad (v_h^E \sigma_{h-1}^E)_{\sigma_g^E} = v_h^E \sigma_{h-1}^E \quad (g \geq h).$$

Applying these relations to (1) just proved, we secure the conclusion in (2). To prove (3), we note that $v_h^E \sigma_h^E (\sigma_f^E) = v_h^E \sigma_{h-1}^E (\sigma_f^E) = v_h^E (\sigma_f^E) \quad (h > f)$. Hence,

$$\begin{aligned} \mu_g^E (\sigma_f^E) &= \sum_h^{ig} \frac{v_h^E \sigma_h^E (\sigma_f^E) - v_h^E \sigma_{h-1}^E (\sigma_f^E)}{2^{h+1}} \\ &= \sum_h^{if} \frac{v_h^E \sigma_h^E (\sigma_f^E) - v_h^E \sigma_{h-1}^E (\sigma_f^E)}{2^{h+1}} = \mu_f^E (\sigma_f^E), \end{aligned}$$

since every term in the summand is equal to a O^2 -vector for $h > f$. To prove (4) and (5), let us introduce the notation

$$\varphi_h^2 \equiv (v_h^2 \sigma_h^2 - v_h^2 \sigma_{h-1}^2) / 2^{h+1}.$$

Then by II A, T I, we have

$$\begin{aligned} M^2 \mu_g^2 &\leq \sum_h^{1g} M^2 \left(\frac{v_h^2 \sigma_h^2 - v_h^2 \sigma_{h-1}^2}{2^{h+1}} \right) \leq \sum_h^{1g} \frac{M^2 v_h^2 \sigma_h^2 + M^2 v_h^2 \sigma_{h-1}^2}{2^{h+1}} \\ &\leq \sum_h^{1g} 1/2^h = 1 - 1/2^g, \end{aligned}$$

whence $\mu_g^2 \uparrow^2$ ($g \geq 1$), and also $\exists \sum_h M^2 \varphi_h^2 \leq 1$. Applying IIB, T VI, we may conclude that

$$\exists \sum_{h^2} \varphi_h^2 = \lim_{g^2} \sum_h^{1g} \varphi_h^2 = \lim_{g^2} \mu_g^2.$$

Denote this limit by μ_*^2 . Then

$$M^2 \mu_*^2 = M^2 \sum_{h^2} \varphi_h^2 \leq \sum_h M^2 \varphi_h^2 \leq 1,$$

and hence $\mu_*^2 \uparrow^2$. Finally, for (6), we note, by (3) just proved, that

$$\mu_*^2(\sigma_f^2) = \lim_g \mu_g^2(\sigma_f^2) = \lim_{g^2} \mu_g^2(\sigma_f^2) = \lim_{g^2} \mu_f^2(\sigma_f^2) = \mu_f^2(\sigma_f^2).$$

iv^0 is now completed.

$$v^0 \quad h \geq 1 \quad \cdot) \cdot \quad M_{\sigma_h}^{1, J^2 K^{12}} \mu_h^2 > h + 1.$$

In fact, by definition of μ_h^2 in iv^0 , it follows that

$$M_{\sigma_h}^{1, J^2 K^{12}} \mu_h^2 \geq M_{\sigma_h}^{1, J^2 K^{12}} \frac{v_h^2 \sigma_h^2}{2^{h+1}} - M_{\sigma_h}^{1, J^2 K^{12}} (\mu_{h-1}^2 - \frac{v_h^2 \sigma_{h-1}^2}{2^{h+1}}).$$

Let us now consider the second term on the right. By iv^0 , (2), (4); II A, T VI, T III, and T I; III A, T II; and 11^0 , we secure

$$\begin{aligned}
M_{\sigma_h}^1 J^2 K^{12} (u_{h-1}^2 - \frac{v_h^2}{2^{h+1}}) &= M_{\sigma_h}^1 J^2 K^{12} (u_{h-1}^2 - \frac{v_h^2}{2^{h+1}}) \\
&\leq M^1 J^2 K^{12} (u_{h-1}^2 - \frac{v_h^2}{2^{h+1}}) \\
&\leq M^1 K^{12} (M^2 u_{h-1}^2 + M^2 \frac{v_h^2}{2^{h+1}}) \\
&\leq 2M^1 K^{12} .
\end{aligned}$$

Hence from iii⁰ and the two inequalities just established, we secure v⁰.

$$v1^0 \quad f \geq 1 \quad \cdot) \cdot \quad M_{\sigma_f}^1 J^2 K^{12} u_*^2 > f.$$

By iv⁰, (2) and (6), we secure

$$J^2 K^{12} u_f^2 = J^2 K^{12} u_{\sigma_f}^2 = J_{\sigma_f}^2 K^{12} u_f^2 = J_{\sigma_f}^2 K^{12} u_*^2 = J^2 K^{12} u_*^2 .$$

Hence by II A, T IV, T I, T VI, and iv⁰, (2), (6), the following results are secured:

$$\begin{aligned}
M_{\sigma_f}^1 J^2 K^{12} u_*^2 &= M_{\sigma_f}^1 (J^2 K^{12} u_f^2 + J^2 (K^{12} - K_{\sigma_f}^{12}) u_*^2) \\
&\geq M_{\sigma_f}^1 J^2 K^{12} u_f^2 - M_{\sigma_f}^1 J^2 (K^{12} - K_{\sigma_f}^{12}) u_*^2 .
\end{aligned}$$

By v⁰ the first term on the right is $> f + 1$, and the second term, by reasoning similar to that in the proof of iii⁰, is ≥ -1 . Thus we have secured v1⁰.

Now by iv⁰, (5), we have $u_*^2 M^2$ and hence by hypothesis, $J^2 K^{12} u_*^2 M^1$. Therefore,

$$M_{\sigma_f}^1 J^2 K^{12} u_*^2 \leq M^1 J^2 K^{12} u_*^2 < \infty \quad (\sigma'),$$

which contradicts the result in v1⁰. The theorem is thus proved.

By III A, T III, 9), we secure

$$\text{Cor.} \quad K^{12} \overline{\text{rows}} M^2 \cdot J^2 K^{12} u_*^2 M^1 (u^2) \sim K^{12} M^{12}$$

§ 2.

$$T\ 12) \quad \epsilon_0^1 \text{ PH} :): \quad 12.1) \quad A'^0 \subset A' \sim \epsilon_0^1 A^{11}$$

$$12.2) \quad (1) \quad \mathcal{M}'^0 \subset \mathcal{M}' \sim (2) \quad \epsilon_0^1 M^{11}_0$$

$$(3) \quad \epsilon_0^1 M^{10}_1 \sim (4) \quad \epsilon_0^1 M^{11}$$

$$12.3) \quad F'^0 \subset F' \sim \epsilon_0^1 F^1 \bar{F}^1$$

The proof for 12.1) and 12.3) are simple, and hence omitted. We shall prove 12.2) by the following scheme:

$$(1) \rightarrow (2) \rightarrow (3) \rightarrow (2).(3) \rightarrow (4) \rightarrow (1).$$

To prove $(1) \rightarrow (2)$, we note that

$$\epsilon_0^1 \overline{\text{rows } M^1} \cdot J^1_0 \epsilon_0^1 \mu^1_0 M^{10} \therefore M^1 (\mu^1_0).$$

Hence by Cor. to T 11), $\epsilon_0^1 M^{11}_0$. (2) is equivalent to (3), since ϵ_0^1 is hermitian. To prove $(2).(3) \rightarrow (4)$, we make use of the composition theorem of modularity in III A, and secure

$$J^1_0 \epsilon_0^1 \epsilon_0^1 = \epsilon_0^1 M^{11}.$$

That $(4) \rightarrow (1)$ is demonstrated in III A, T I.

For convenience, $\mathcal{K}^{12}$ will indicate the class of K^{12} modular relative to $\epsilon^1 \epsilon^2$.

$$T\ 13) \quad (\epsilon_0^1 \epsilon_0^2)^{\text{PH}} \cdot \mathcal{K}'^{12}_0 \cdot \mathcal{K}'^{12} \cdot \mathcal{F} \equiv [\text{all } K^{12} T^{12}] :):$$

$$(1) \quad \mathcal{K}'^{12}_0 \subset \mathcal{F}'^{12} \sim (2) \quad \epsilon_0^1 M^{11} \cdot \epsilon_0^2 M^{22} \sim (3) \quad \mathcal{K}'^{12}_0 \subset \mathcal{K}'^{12}.$$

To prove this theorem, we shall adopt the following scheme:

$$(1) \rightarrow (2) \rightarrow (3) \rightarrow (1).$$

That (2) implies (3) follows from III A, T I, 15). Also (3) implies (1) since $\mathcal{K}'^{12} \subset \mathcal{F}'^{12}$. It remains to show that (1) implies

(2). If we prove $\mathcal{M}' \subset \mathcal{M}^1$, $\mathcal{M}^{20} \subset \mathcal{M}^2$, then by T 12.2) the theorem is established. Now consider any $\mu^{10} \cdot \mu^{20} \neq 0^2$. Define $K^{12} \equiv \mu^{10} \overline{\mu^{20}}$. Then by III A, T IV, $K^{12} M^{10} \mu^{20}$, which, by hypothesis, gives $K^{12} T^{12}$. Since $\mu^{20} \neq 0^2$, then $\exists p^2 \ni \mu^{20}(p^2) \neq 0$. Thus $\mu^{10} \overline{\mu^{20}(p^2)} M^1$, and hence $\mu^{10} M^1$. Similarly, $\mathcal{M}^{20} \subset \mathcal{M}^2$.

$$D 11) K^{12} T^{12} :) : 1) K^{12} C_2^{12} := \mathcal{L} R^{TC} \cdot (\varphi_l^2 \mid l) \cdot \mu^2 \ni \frac{1}{l^2} \varphi_l^2 = \mu^1 \cdot) .$$

$$\frac{1}{l^2} J^2 K^{12} \varphi_l^2 = J^2 K^{12} \mu^2 .$$

$$2) K^{12} C_1^{12} := \mathcal{L} R^{TC} \cdot (\varphi_l^2 \mid l) \cdot \mu^2 \ni \frac{1}{l^2} \varphi_l^2 = \mu^1 \cdot) .$$

$$\frac{1}{l^2} J^2 K^{12} \varphi_l^2 = J^2 K^{12} \mu^2 .$$

$$T 14.1) K^{12} T^{12} :) : C_1^{12} \cdot) \cdot C_2^{12}$$

$$T 14.2) K^{12} T^{12} \cdot C_2^{12} \cdot) \cdot K^{12} M^{12}$$

14.1) is evident from the fact that mode 2 convergence implies mode 1. To prove 14.2) we observe that since $\lim_{\sigma^2 \rightarrow 2^2} \mu_{\sigma^2}^2 = \mu^2$ for every μ^2 , we have from C_2^{12}

$$\mu^2 \cdot) \cdot \lim_{\sigma^2 \rightarrow 1^2} J^2 K^{12} \mu_{\sigma^2}^2 = J^2 K^{12} \mu^2 ,$$

from which it follows that $\mu^2 \cdot) \cdot J^2 K^{12} \mu^2 M^1$. Thus by the corollary of the generalized Hellinger-Toeplitz theorem, $K^{12} M^{12}$.

Here we establish the modularity of K^{12} from its continuity property without using any notion of bilinear functional theory. This theorem has no direct application in the future development, nevertheless, it tells us that some continuous conditions can never be imposed upon $K^{12} T^{12}$ without strengthening it to be M^{12} .

II. ON T-MATRICES AND THEIR ASSOCIATED SPACES

E. H. Moore in his second theory of General Analysis developed a theory concerning a matrix K^{12} whose columns are modular functions relative to ε^1 of the basis $(\mathcal{A}^D, \mathcal{P}', \mathcal{P}^2, \varepsilon^1 \text{ PH})$. From the given matrix K^{12} , a positive and hermitian matrix $\varepsilon_\kappa^2 \equiv J^{1\kappa} K^{*21} K^{12}$ is derived. Then K^{12} is proved to be modular relative to $(\varepsilon^1, \varepsilon_\kappa^2)$, and hence K^{*21} is by columns $M^{2\kappa}$. Applying the same process to K^{*21} with $(\mathcal{A}^2, \mathcal{P}', \mathcal{P}^2, \varepsilon_\kappa^2)$ as basis, we obtain another positive, hermitian matrix $\varepsilon_\kappa^1 = J^{2\kappa} K^{12} K^{*21}$. Hence K^{*21} is modular relative to $(\varepsilon_\kappa^2, \varepsilon_\kappa^1)$. The matrix ε_κ^1 is modular relative to $(\varepsilon^1, \varepsilon^1)$, and hence $\mathcal{M}'_\kappa \subset \mathcal{M}'$. One of the important properties of ε^1 is that $J^{1\kappa} = J^1$ as on $\overline{\mathcal{M}'_\kappa} \mathcal{M}'_\kappa$, from which we deduce immediately $\varepsilon_\kappa^2 = J^{1\kappa} K^{*21} K^{12}$.

If K^{12} is also by rows M^2 relative to a given positive, hermitian matrix ε^2 , we may apply the preceding theory to K^{*21} and obtain $\varepsilon_{\kappa^*}^1, \varepsilon_{\kappa^*}^2$ and their corresponding spaces $\mathcal{M}'_{\kappa^*}, \mathcal{M}^{2\kappa^*}$.

As a matter of fact, we have two (1-1) correspondances,

$$\begin{aligned} \mathcal{M}'_\kappa &\sim \mathcal{M}^{2\kappa} \text{ via } J^{1\kappa} K^{*21}, J^{2\kappa} K^{12} \\ \mathcal{M}^{2\kappa^*} &\sim \mathcal{M}'_{\kappa^*} \text{ via } J^{2\kappa^*} K^{12}, J^{1\kappa^*} K^{*21} \end{aligned}$$

In case K^{12} is M^{12} , then $\mathcal{M}'_{\kappa^*} \subset \mathcal{M}'_\kappa$, $\mathcal{M}^{2\kappa} \subset \mathcal{M}^{2\kappa^*}$.

The theory of IV A is developed with the idea of generalizing Hilbert's theory of limited quadratic forms of infinitely many variables. However, for the study of non-modular matrices, the theory as developed in IV A is not sufficient. Consider, in particular, the problem of reciprocals. In the case $K^{12} M^{12}$, using Moore's definition of reciprocal λ^{21} in IV C, we obtain the

following equivalent definitions:

$$\begin{aligned} \lambda^{21} M^{21} T^{2k*1k} : J^2 K^{12} p^{2k*} = p^{1k*} \quad .) \quad J^1 \lambda^{21} p^{1k*} = p^{2k*} \\ : J^1 K^{*21} p^{1k} = p^{2k} \quad .) \quad J^2 \lambda^{*12} p^{2k} = p^{1k} \end{aligned}$$

By comparison of the last two conditions with results from IV A concerning $\mathcal{M}^{1k*}$, $\mathcal{M}^{2k*}$, and $\mathcal{M}^{2k}$, $\mathcal{M}^{1k}$, viz.

$$\begin{aligned} J^2 K^{12} p^{2k*} = p^{1k*} \quad .\sim \quad J^{1k*} K^{*21} p^{1k*} = p^{2k*}, \\ J^{1k*} K^{*21} p^{1k} = p^{2k} \quad .\sim \quad J^{2k} K^{12} p^{2k} = p^{1k} \quad , \end{aligned}$$

we notice that

$$\begin{aligned} J^{1k*} K^{*21} = J^1 \lambda^{21} \quad \text{on } \mathcal{M}^{1k*} \text{ to } \mathcal{M}^{2k*}, \\ J^{2k} K^{12} = J^2 \lambda^{*12} \quad \text{on } \mathcal{M}^{2k} \text{ to } \mathcal{M}^{1k}. \end{aligned}$$

In case $K^{12} T^{12}$, whether the reciprocal λ^{21} is assumed to be M^{12} or only of T^{12} , the transformations $J^1 \lambda^{21}$ and $J^2 \lambda^{*12}$ are not well defined for every p^{1k*} and p^{2k} respectively, since in general $\mathcal{M}^{1k*} \subset \mathcal{M}^1$, $\mathcal{M}^{2k} \subset \mathcal{M}^2$. In order that the transformations be well defined, the vectors p^{1k*} and p^{2k} must be restricted to certain subspaces in $\mathcal{M}^{1k*}$ and $\mathcal{M}^{2k}$ respectively. In other words, the vectors p^{2k*} and p^{1k} in the transformations $J^2 K^{12}$ and $J^1 K^{*21}$ must be restricted to certain subspaces in $\mathcal{M}^{2k*}$ and $\mathcal{M}^{1k}$ respectively.

In the present section, we shall develop some properties of $K^{12} T^{12}$ which focus on the problems concerning (1) the subspaces in $\mathcal{M}^2$ and $\mathcal{M}^1$ that can be transformed by $J^2 K^{12}$ and $J^1 K^{*21}$ into new subspaces of $\mathcal{M}^1$ and $\mathcal{M}^2$ respectively, and (2) the interchange of the integration processes J^1 and J^2 .

In abbreviation the basis for the present section will

be denoted by

$$\alpha^D \cdot \vartheta' \cdot \vartheta^2 \cdot \epsilon' \cdot \epsilon^2 \cdot \kappa'^2 \cdot T'^2$$

§ 3

Since $K^{12} T^{12}$, we may make an application of IV A both to K^{12} and to K^{21} , obtaining $\epsilon_K^2 \cdot \epsilon_K^1 \cdot \epsilon_{K^*}^1 \cdot \epsilon_{K^*}^2$, whence the operators J^{2K} , J^{1K} , J^{1K^*} , J^{2K^*} and the spaces $\mathcal{M}^{2K}$, $\mathcal{M}^{1K}$, $\mathcal{M}^{1K^*}$, $\mathcal{M}^{2K^*}$. The first theorem studies the interrelation between $\mathcal{M}^1$, $\mathcal{M}^2$ and the four spaces just mentioned. Further, in this theorem, we show which of the nine modularities for K^{12} and the six ϵ 's are always present and finally we derive necessary and sufficient conditions that the remaining modularities should be present.

For convenience, we shall adopt the notation $\mathcal{M}'_{\cap(1K'K^*)}$ for $\cap(\mathcal{M}^{1K}, \mathcal{M}^{1K^*})$ and similarly for other cases.

T 21) 21.0) $\epsilon^1 M^{11}$ and in general has no other modularities.

$$\begin{aligned} 21.1) \quad M^{11} \epsilon_K^1 &= M^{1K} \epsilon_K^1 = M^{11K} \epsilon_K^1 = M^{1K} \epsilon_K^1 = 0 \quad (K^{12} = O^{12}) \\ &= 1 \quad (K^{12} \neq O^{12}) \end{aligned}$$

$$\begin{aligned} 21.2) \quad M^{12K} K^{12} &= M^{1K} \epsilon_K^{12} K^{12} = M^{1K^*2} K^{12} = M^{1K^*2K^*} K^{12} = 0 \quad (K^{12} = O^{12}) \\ &= 1 \quad (K^{12} \neq O^{12}) \end{aligned}$$

21.3) $\epsilon_{K^*}^1 M^{1K^*1K^*}$ and in general has no other modularities.

$$21.41) \quad \mathcal{M}^1 \supset \mathcal{M}^{1K}$$

$$21.42) \quad \mathcal{M}^{1K^*} = J^2 K^{12} \mathcal{M}^2$$

$$21.43) \quad [\text{cols } K^{12}]_L \subset \mathcal{M}'_{\cap(1K'K^*)} \subset \mathcal{M}'_{\cap(1K'K^*)}$$

In general, for $K^{12} T^{12}$, the space $\mathcal{M}'_{\cap(1K'K^*)}$ is not identical with either $\mathcal{M}^{1K}$ or $\mathcal{M}^{1K^*}$, as examples show. (A similar re-

mark applies to $\mathcal{M}'_{\alpha(1, 'k*)}$.) In case $\mathcal{M}'_{\alpha(1, 'k*)}$ is identical with either $\mathcal{M}'_{k*}$ or $\mathcal{M}'_k$, then K^{12} must have additional properties, which will be studied in 21.5) and 21.6).

$$21.5) \quad (1) \quad \varepsilon_{k*}^1 \text{ all 9 mods } \sim. \quad (2) \quad \varepsilon_{k*}^1 \text{ has any 2nd mod } \sim. \\ (3) \quad K^{12} M^{12} \sim. \quad (4) \quad K^{12} M^{1k*2} \sim. \quad (5) \quad K^{12} M^{1k*2k*} \sim.$$

$$(6) \quad \mathcal{M}'_k \supset \mathcal{M}'_{k*} \sim. \quad (7) \quad \mathcal{M}'_{\alpha(1, 'k*)} = \mathcal{M}'_{k*} \sim.$$

$$(8) \quad \mathcal{M}' \supset \mathcal{M}'_{k*} \sim. \quad (9) \quad \mathcal{M}'_{\alpha(1, 'k*)} = \mathcal{M}'_{k*}.$$

$$21.6) \quad (1) \quad \varepsilon_k^1 \text{ all 9 mods } \sim. \quad (2) \quad \varepsilon_k^1 \text{ has any 5th mod } \sim.$$

$$(3) \quad \mathcal{M}'_k \subset \mathcal{M}'_{k*} \sim. \quad (4) \quad \mathcal{M}'_{\alpha(1, 'k*)} \text{ LJ}^1 \sim. \quad (5) \quad K^{12} M^{1k*2k}$$

$$21.7) \quad K^{12} M^{12} \cdot). \quad M^{1k*2} K^{12} = M^{12k*} K^{12} = M^{1k*2k*} K^{12} = M^{12} K^{12}$$

$$21.8) \quad K^{12} M^{1k*2k} \cdot). \quad (1) \quad M^{11k*} \varepsilon_k^1 = M^{1k*1k*} \varepsilon_k^1 = M^{1k*1} \varepsilon_k^1$$

$$= M^{1k*1k} \varepsilon_k^1 = M^{1k*2k} K^{12}$$

$$(2) \quad M^{1k*1k*} \varepsilon_k^1 = N^{1k*2k} K^{12}$$

$$\text{GT. 21)} \quad K^{12} M^{12} :): K^{12} M^{1k*2k} \sim. \quad \mathcal{M}'_{k*} \text{ LJ}^1$$

The statements 21.0) and 21.3) are obvious; 21.1) and 21.2) follow from IV A theory.

In 21.4), we note the first two parts follow from IV A. That $\mathcal{M}'_{\alpha(1, 'k*)} \supset \mathcal{M}'_{\alpha(1, 'k*)}$ is evident from 21.41). By hypothesis, $K^{12} T^{12}$, and hence from IV A theory, $K^{12} T^{1k*2}$, T^{1k*2} . Thus $[\text{cols } K^{12}] \subset \cap (\mathcal{M}'_k, \mathcal{M}'_{k*}) \equiv \mathcal{M}'_{\alpha(1, 'k*)}$. Now it is evident that both $\mathcal{M}'_{\alpha(1, 'k*)}$ and $\mathcal{M}'_{\alpha(1, 'k*)}$ are linear. Hence we have 21.43).

To prove 21.5), we first show that

$$\varepsilon_{k*}^1 M^{11} \rightarrow (3) \rightarrow (4) \rightarrow (5) \rightarrow \varepsilon_{k*}^1 M^{1k*1k} \rightarrow \varepsilon_{k*}^1 M^{11}.$$

That $\varepsilon_{K^*}^1 M^{11} \rightarrow (3)$ follows immediately from III A, T III and the definition of $\varepsilon_{K^*}^1$. For $(3) \rightarrow (4)$, we note from IV A that $\varepsilon_K^1 M^{1K^*1}$ and $K^{12} = J^1 \varepsilon_K^1 K^{12}$. Thus by the composition theorem of modularity we secure (4) . That $(4) \rightarrow (5)$ follows from the fact that $\varepsilon_{K^*}^2 M^{22K^*}$ and $K^{12} = J^2 K^{12} \varepsilon_{K^*}^2$. To prove (5) implies $\varepsilon_{K^*}^1 M^{1K^*1K^*}$, we use the relation $J^{2K^*} K^{12} K^{*21} = \varepsilon_{K^*}^1$, and the composition theorem of modularity. That $\varepsilon_{K^*}^1 M^{1K^*1K^*} \rightarrow \varepsilon_{K^*}^1 M^{11}$ is obvious, since ε_K^1 is M^{11} .

By 12.2), we obtain at once the following:

$$\begin{aligned} \varepsilon_{K^*}^1 M^{1K^*1K^*} &\sim \varepsilon_{K^*}^1 M^{1K^*1K^*} \sim \varepsilon_{K^*}^1 M^{1K^*1K^*}, \\ \varepsilon_{K^*}^1 M^{11} &\sim \varepsilon_{K^*}^1 M^{11K^*} \sim \varepsilon_{K^*}^1 M^{1K^*1}. \end{aligned}$$

Furthermore, since ε_K^1 is M^{11} , we have by 12.3) that

$$\varepsilon_{K^*}^1 M^{1K^*1K^*} \rightarrow \varepsilon_{K^*}^1 M^{1K^*1} \rightarrow \varepsilon_{K^*}^1 M^{11K^*} \rightarrow \varepsilon_{K^*}^1 M^{11}.$$

Now, that $\varepsilon_K^1 M^{11}$ implies $\varepsilon_{K^*}^1 M^{1K^*1K^*}$ is proved above, and hence any one of the eight modularities for $\varepsilon_{K^*}^1$ is equivalent to any other. Since (1) implies (2) is evident, we secure that (1) is equivalent to (2). Thus we establish the equivalence of the first five properties in 21.5). The balance of 21.5) follows from 12.2).

The scheme for the proof of 21.6) will be as follows:

$$(1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (5) \rightarrow (1).$$

That (1) implies (2) is obvious. By 12.2), and $\varepsilon_K^1 M^{11}$, we secure

$$\begin{aligned} \varepsilon_K^1 M^{1K^*1K^*} \rightarrow \varepsilon_K^1 M^{1K^*1K^*} \sim \varepsilon_K^1 M^{1K^*1K^*} \rightarrow \varepsilon_K^1 M^{11K^*} \sim \varepsilon_K^1 M^{1K^*1} \\ \rightarrow \varepsilon_K^1 M^{1K^*1K^*} \sim \mathcal{M}'_{K^*} \subset \mathcal{M}'_{K^*}. \end{aligned}$$

This proves that (2) implies (3). To show (3) implies (4), we note that $\mathcal{M}'_{K^*} \subset \mathcal{M}'_{K^*}$ gives $\mathcal{M}'_{\mathcal{M}'_{K^*}} = \mathcal{M}'_{K^*} L J^1$, and hence (4). Now $\mathcal{M}'_{\mathcal{M}'_{K^*}} L J^1$ gives, by virtue of IV B, T V, that

$$\varphi^{12} \text{ cols } \pi'_{\alpha('_{\kappa} '_{\kappa*})} \cdot \pi'_{\varepsilon_{\varphi}} \subset \pi'_{\alpha('_{\kappa} '_{\kappa*})}.$$

But K^{12} cols $\pi'_{\alpha('_{\kappa} '_{\kappa*})}$, and consequently, $\pi'_{\kappa} \subset \pi'_{\alpha('_{\kappa} '_{\kappa*})} \subset \pi'_{\kappa*}$. Therefore, by 12.2), $\varepsilon_{\kappa}^1 M^{1_{\kappa*} 1_{\kappa*}}$. It follows from the relation $K^{12} = J^{1_{\kappa}} \varepsilon_{\kappa}^1 K^{12}$ and $K^{12} M^{1_{\kappa} 2_{\kappa}}$, that $K^{12} M^{1_{\kappa*} 2_{\kappa}}$. The statement (5) .). (1) is secured from $\varepsilon_{\kappa}^1 = J^{2_{\kappa}} K^{12} K^{*21}$ and the composition theorem of modularity, since $K^{12} M^{12_{\kappa}} \cdot M^{1_{\kappa} 2_{\kappa}} \cdot M^{1_{\kappa*} 2_{\kappa}}$.

21.7) follows from IV A, T II.

If $K^{12} = O^{12}$, the results in 21.8) are obvious. Assuming that $K^{12} \neq O^{12}$, we shall show that

$$M^{1_{\kappa*} 2_{\kappa}} K^{12} \leq M^{1_{\kappa*} 1} \varepsilon_{\kappa}^1 = M^{1_{\kappa*} 1_{\kappa}} \varepsilon_{\kappa}^1 \leq M^{1_{\kappa*} 2_{\kappa}} K^{12}.$$

From $K^{12} = J^1 \varepsilon_{\kappa}^1 K^{12}$, we obtain, by III A, T II, 7) that $M^{1_{\kappa*} 2_{\kappa}} K^{12} \leq M^{1_{\kappa*} 1} \varepsilon_{\kappa}^1 M^{12_{\kappa}} K^{12} = M^{1_{\kappa*} 1} \varepsilon_{\kappa}^1$, for $M^{12_{\kappa}} K^{12} = 1$. That $M^{1_{\kappa*} 1} \varepsilon_{\kappa}^1 = M^{1_{\kappa*} 1_{\kappa}} \varepsilon_{\kappa}^1$ follows from IV A, T II. Finally, we use the relation $\varepsilon_{\kappa}^1 = J^{2_{\kappa}} K^{12} K^{*21}$ and secure

$$M^{1_{\kappa} 1_{\kappa*}} \varepsilon_{\kappa}^1 \leq M^{1_{\kappa} 2_{\kappa}} K^{12} M^{2_{\kappa} 1_{\kappa*}} K^{*21} = M^{1_{\kappa*} 2_{\kappa}} K^{12},$$

for $M^{1_{\kappa} 2_{\kappa}} K^{12} = 1$.

To prove the equality in (2), we first show that $M^{1_{\kappa*} 1_{\kappa*}} \varepsilon_{\kappa}^1 \leq N^{1_{\kappa*} 2_{\kappa}} K^{12}$. Now, $\varepsilon_{\kappa}^1 = J^{1_{\kappa}} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1$. Hence, by III A, T II, 7) and (1) just proved, we secure $M^{1_{\kappa*} 1_{\kappa*}} \varepsilon_{\kappa}^1 \leq (M^{1_{\kappa*} 1_{\kappa}} \varepsilon_{\kappa}^1)^2 \leq N^{1_{\kappa*} 2_{\kappa}} K^{12}$. Next, we shall show that $M^{1_{\kappa*} 1_{\kappa*}} \varepsilon_{\kappa}^1 \geq N^{1_{\kappa*} 2_{\kappa}} K^{12}$. By virtue of III A, T I, 15), we have

$$M^{1_{\kappa*} 2_{\kappa}} K^{12} \leq M^{1_{\kappa} 2_{\kappa}} K^{12} (M^{1_{\kappa*} 1_{\kappa*}} \varepsilon_{\kappa}^1)^{\frac{1}{2}}.$$

Since $M^{1_{\kappa} 2_{\kappa}} K^{12} = 1$, we obtain the required inequality by squaring both sides. This completes the proof of theorem 21).

From this theorem, we learn that certain conditions can

never be imposed upon K^{12} without making it modular relative to $(\varepsilon^1 \ \varepsilon^2)$ or to $(\varepsilon_{\kappa}^1 \ \varepsilon_{\kappa}^2)$.

§ 4

Since $K^{12} T^{12}$, it follows from IV A and III A that K^{12} is also by cols $\mathcal{M}^{\kappa}$, $\mathcal{M}^{\kappa*}$ and by rows $\overline{\mathcal{M}^{2\kappa*}}$, $\mathcal{M}^{2\kappa}$. From the preceding theorem, we find that $K^{12} M^{12} \sim \mathcal{M}^{\kappa*} < \mathcal{M}^1 \sim \mathcal{M}^{\kappa*} < \mathcal{M}^{\kappa}$. In other words, not every Fourier coefficient function of μ^2 relative to K^{*21} is a vector in the $\mathcal{M}^1$ or $\mathcal{M}^{\kappa}$ space. In order that $J^2 K^{12} \mu^2$ be a vector in $\mathcal{M}^1$ or $\mathcal{M}^{\kappa}$, the vector μ^2 must be restricted to a subspace in $\mathcal{M}^2$. Theorem 22) is devoted to a study of this problem. For convenience, we shall adopt the following notations:

$$D \ 21) \quad \mathcal{M}^{2\kappa 1} \equiv [\mu^{2\kappa 1}] \equiv [\text{all } \mu^2 \ni J^2 K^{12} \mu^2 M^1]$$

$$D \ 22) \quad \mathcal{M}_{\kappa}^{1 \cdot 2} \equiv [J^2 K^{12} \mu^{2\kappa 1} \mid \mu^{2\kappa 1}]$$

It is evident from definitions 21) and 22) that $\mathcal{M}^{2\kappa 1} < \mathcal{M}^2$, and $\mathcal{M}_{\kappa}^{1 \cdot 2} < \mathcal{M}^1$. Also, every $\mu^2 \in O^2[\text{rows } K^{*21}]$ is in $\mathcal{M}^{2\kappa 1}$.

Since K^{12} has all of the eight additional types, four of which are $T^{1\kappa 2}$, $T^{12\kappa*}$, $T^{1\kappa 2\kappa*}$, and $T^{1\kappa* 2\kappa}$, we have as important instances of the definition 21) the spaces

$$\mathcal{M}^{2\kappa 1\kappa}, \quad \mathcal{M}^{2\kappa*\kappa 1}, \quad \mathcal{M}^{2\kappa*\kappa 1\kappa}, \quad \mathcal{M}^{2\kappa\kappa 1\kappa*},$$

from which we obtain as instances of D 22) the spaces

$$\mathcal{M}_{\kappa}^{1\kappa' 2}, \quad \mathcal{M}_{\kappa}^{1\kappa' 2\kappa*}, \quad \mathcal{M}_{\kappa}^{1\kappa' 2\kappa*}, \quad \mathcal{M}_{\kappa}^{1\kappa* \cdot 2\kappa}$$

respectively. Corresponding to these ten spaces there are ten more associated in a similar fashion with K^{12} which may be written from the above ten by interchanging 1 and 2, also K and K^* .

In order to indicate clearly the numerous one to one correspondences in the sequel, we introduce the following notation:

$$D\ 23) \quad \mathcal{M}_0^1 \cdot \mathcal{M}_0^2 \cdot \varphi^{12} \cdot \psi^{21} \ni J^1 \psi^{21} \text{ on } \mathcal{M}_0^1 \text{ to } \mathcal{M}_0^2, J^2 \varphi^{12} \text{ on } \mathcal{M}_0^2 \text{ to } \mathcal{M}_0^1$$

$$:): \mathcal{M}_0^1 \left(\frac{1}{2} \Psi \right) \mathcal{M}_0^2 \cdot \equiv \cdot \mathcal{M}_0^1 \sim \mathcal{M}_0^2 \quad \text{via } J^1 \psi^{21} \cdot J^2 \varphi^{12}.$$

$$T\ 22) \quad 22.1) \quad K^{12} \ M^{12} \cdot \sim \cdot \mathcal{M}^{2 \times 1} = \mathcal{M}^2 \cdot \sim \cdot \mathcal{M}^{2 \times 1_K} = \mathcal{M}^2$$

$$\cdot \sim \cdot \mathcal{M}^{2_{K^*} \times 1} = \mathcal{M}^{2_{K^*}} \cdot \sim \cdot \mathcal{M}^{2_{K^*} \times 1_K} = \mathcal{M}^{2_{K^*}}$$

$$\cdot \mathcal{M}_K^{1 \cdot 2} = \mathcal{M}_K^{1_{K^*}} \cdot \sim \cdot \mathcal{M}_K^{1_{K^*} \cdot 2} = \mathcal{M}_K^{1_{K^*}} \cdot \sim \cdot \mathcal{M}_K^{1 \cdot 2_{K^*}} = \mathcal{M}_K^{1_{K^*}}$$

$$\cdot \sim \cdot \mathcal{M}_K^{1_{K^*} \cdot 2_{K^*}} = \mathcal{M}_K^{1_{K^*}} \cdot \sim \cdot \mathcal{M}_K^{2_{K^*} \times 1_K} \ L^2$$

$$22.2) \quad K^{12} \text{ comp cols } \mathcal{M}^1 \cdot \cdot \mathcal{M}^{2 \times 1} = \mathcal{M}^{2 \times 1_K} \cdot \mathcal{M}^{2_{K^*} \times 1} = \mathcal{M}^{2_{K^*} \times 1_K}$$

$$\cdot \mathcal{M}_K^{1 \cdot 2} = \mathcal{M}_K^{1_{K^*} \cdot 2} \cdot \mathcal{M}_K^{1 \cdot 2_{K^*}} = \mathcal{M}_K^{1_{K^*} \cdot 2_{K^*}}$$

$$22.31) \quad \mathcal{M}^{2 \times 1} \supset \mathcal{M}^{2 \times 1_K} \supset \mathcal{M}^{2_{K^*} \times 1_K} \cdot \mathcal{M}^{2 \times 1} \supset \mathcal{M}^{2_{K^*} \times 1} \supset \mathcal{M}^{2_{K^*} \times 1_K}$$

$$22.32) \quad \mathcal{M}^{2_{K^*} \times 1} = J^2 \mathcal{E}_{K^*}^2 \mathcal{M}^{2 \times 1}$$

$$22.33) \quad \mathcal{M}^{2_{K^*} \times 1_K} = J^2 \mathcal{E}_{K^*}^2 \mathcal{M}^{2 \times 1_K}$$

$$22.41) \quad \mu^{2 \times 1} \cdot \cdot \cdot \mathbb{E}(\mu^{2_{K^*} \times 1} \cdot \mu_0^2 \mathcal{M}^{2_{K^*}}) \ni \mu^{2 \times 1} = \mu^{2_{K^*} \times 1} + \mu_0^2$$

$$22.42) \quad \mu^{2 \times 1_K} \cdot \cdot \cdot \mathbb{E}(\mu^{2_{K^*} \times 1_K} \cdot \mu_0^2 \mathcal{M}^{2_{K^*}}) \ni \mu^{2 \times 1_K} = \mu^{2_{K^*} \times 1_K} + \mu_0^2$$

$$22.51) \quad \mathcal{M}^{1_{K^*}} \supset \mathcal{M}_K^{1 \cdot 2} = \mathcal{M}_K^{1 \cdot 2_{K^*}} = \mathcal{M}_{\cap(1 \cdot 1_{K^*})}^1 \supset \mathcal{M}_K^{1_{K^*} \cdot 2} = \mathcal{M}_K^{1_{K^*} \cdot 2_{K^*}} = \mathcal{M}_{\cap(1_{K^*} \cdot 1_{K^*})}^1$$

$$22.52) \quad \mathcal{M}^{2_{K^*} \times 1} = J^{1_{K^*}} K^{*21} \mathcal{M}_{\cap(1 \cdot 1_{K^*})}^1$$

$$22.53) \quad K^{12} \text{ comp rows } \mathcal{M}^2 \cdot \sim \cdot \mathcal{M}^{2 \times 1} = J^{1_{K^*}} K^{*21} \mathcal{M}_{\cap(1 \cdot 1_{K^*})}^1$$

$$22.54) \quad \mathcal{M}^{2_{K^*} \times 1_K} = J^{1_{K^*}} K^{*21} \mathcal{M}_{\cap(1_{K^*} \cdot 1_{K^*})}^1$$

$$22.55) \quad \mathcal{M}^{2_{K^*} \times 1_K} \left(\begin{smallmatrix} 2_{K^*} & 1_K \\ 1_{K^*} & K^* \end{smallmatrix} \right) \mathcal{M}_{\cap(1_{K^*} \cdot 1_{K^*})}^1$$

$$\begin{aligned}
22.56) \quad (1) \quad \mathfrak{M}_{\alpha(\cdot)'_{\kappa^*}}^1 &= \mathfrak{M}_{\alpha(\cdot)'_{\kappa^*}}^1 \sim. \quad (2) \quad \mathfrak{M}_{\alpha(\cdot)'_{\kappa^*}}^1 \subset \mathfrak{M}^1_{\kappa^*} \\
&\sim. \quad (3) \quad \mathfrak{M}_{\kappa^*}^{1 \cdot \mathfrak{s}} = \mathfrak{M}_{\kappa^*}^{1 \cdot \kappa \cdot \mathfrak{s}} \sim. \quad (4) \quad \mathfrak{M}^{\mathfrak{s} \times 1} = \mathfrak{M}^{\mathfrak{s} \times 1}_{\kappa} \\
&\sim. \quad (5) \quad \mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1} = \mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa} \\
&\sim. \quad (6) \quad J^1 \overline{\varphi^1} J^{\mathfrak{s}} K^{1 \mathfrak{s}} \mathfrak{u}^{\mathfrak{s} \times 1} = J^{\mathfrak{s}} (J^1 \overline{\varphi^1} K^{1 \mathfrak{s}}) \mathfrak{u}^{\mathfrak{s} \times 1} \quad (\varphi^1_{\kappa} F^{1 \kappa} \cdot \mathfrak{u}^{\mathfrak{s} \times 1}) \\
&\sim. \quad (7) \quad J^1 \overline{\varphi^1} J^{\mathfrak{s}} K^{1 \mathfrak{s}} \mathfrak{u}^{\mathfrak{s} \times \kappa \cdot 1} = J^{\mathfrak{s}} (J^1 \overline{\varphi^1} K^{1 \mathfrak{s}}) \mathfrak{u}^{\mathfrak{s} \times \kappa \cdot 1} \\
&\quad \quad \quad (\varphi^1_{\kappa} F^{1 \kappa} \cdot \mathfrak{u}^{\mathfrak{s} \times \kappa \cdot 1})
\end{aligned}$$

To prove 22.1), let us prove $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}} \sim. \mathfrak{M}^{\mathfrak{s} \times 1} = \mathfrak{M}^{\mathfrak{s}}$ and $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}} \sim. \mathfrak{M}^{1 \kappa^*} = \mathfrak{M}_{\kappa^*}^{1 \cdot \mathfrak{s}}$, as the balance, except the last statement, can be shown in a similar manner using 21.5). Evidently, $\mathfrak{M}^{\mathfrak{s} \times 1} \subset \mathfrak{M}^{\mathfrak{s}}$. By III A, T II, 5), we have $J^{\mathfrak{s}} K^{1 \mathfrak{s}} \mathfrak{u}^{\mathfrak{s}} M^{1 \mathfrak{s}}$ for every $\mathfrak{u}^{\mathfrak{s}}$, whence $\mathfrak{M}^{\mathfrak{s}} \subset \mathfrak{M}^{\mathfrak{s} \times 1}$. Thus $\mathfrak{M}^{\mathfrak{s}} = \mathfrak{M}^{\mathfrak{s} \times 1}$. Conversely, if $J^{\mathfrak{s}} K^{1 \mathfrak{s}} \mathfrak{u}^{\mathfrak{s}} M^{1 \mathfrak{s}}$ for every $\mathfrak{u}^{\mathfrak{s}}$, then, by virtue of the corollary to the Hellinger-Toeplitz theorem, $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}}$. Hence $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}}$ is equivalent to $\mathfrak{M}^{\mathfrak{s}} = \mathfrak{M}^{\mathfrak{s} \times 1}$. Now, by the fact just proved, D 22, and 21.42), we secure

$$\mathfrak{M}_{\kappa^*}^{1 \cdot \mathfrak{s}} = [J^{\mathfrak{s}} K^{1 \mathfrak{s}} \mathfrak{u}^{\mathfrak{s}} | \mathfrak{u}^{\mathfrak{s}}] = \mathfrak{M}^{1 \kappa^*}.$$

Conversely, if $\mathfrak{M}^{1 \kappa^*} = \mathfrak{M}_{\kappa^*}^{1 \cdot \mathfrak{s}}$, then $\mathfrak{M}^{1 \kappa^*} \subset \mathfrak{M}^1$. Hence $K^{1 \mathfrak{s}}$ is $M^{1 \mathfrak{s}}$ by 21.5). This proves that $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}} \sim. \mathfrak{M}^{1 \kappa^*} = \mathfrak{M}_{\kappa^*}^{1 \cdot \mathfrak{s}}$.

That $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}}$ implies $\mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa} L J^{\mathfrak{s}}$ follows from the fact that $\mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa} = \mathfrak{M}^{\mathfrak{s} \times \kappa} L J^{\mathfrak{s}}$ when $K^{1 \mathfrak{s}} M^{1 \mathfrak{s}}$. To prove the converse, let $K_{\sigma^2}^{* \mathfrak{s} 1} \equiv J_{\sigma^2}^{\mathfrak{s} \kappa^*} \epsilon_{\kappa^*}^{* \mathfrak{s} 1} K^{* \mathfrak{s} 1}$. Then for every p^1 , $(K_{\sigma^2}^{* \mathfrak{s} 1}(\cdot p^1) | \sigma^{\mathfrak{s}})$ is contained in $\mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa}$, and

$$\bigcup_{\sigma^2} K_{\sigma^2}^{* \mathfrak{s} 1}(\cdot p^1) = \bigcup_{\sigma^2} K_{\sigma^2}^{* \mathfrak{s} 1}(\cdot p^1) = K^{* \mathfrak{s} 1}(\cdot p^1) \quad (p^1).$$

Since by hypothesis, $\mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa}$ is $L J^{\mathfrak{s}}$, we secure $K^{1 \mathfrak{s}} \overline{\text{rows}} \mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa}$. Thus by IV B, T V, we secure $\mathfrak{M}^{\mathfrak{s} \times \kappa^*} \subset \mathfrak{M}^{\mathfrak{s} \times \kappa \cdot 1}_{\kappa}$. Consequently, we

have $\mathcal{M}^{2_{K^*} \times 1_K} = \mathcal{M}^{2_{K^*}}$ and K^{12} is M^{12} .

22.2) is an immediate corollary of IV A, T III.

22.31) is obvious from D 21).

To prove 22.32), we note that

$$\begin{aligned} \mu^{2_{K^*} \times 1} \cdot (J^2 \epsilon_{K^*}^2 \mu^{2_{K^*} \times 1}) M^{2_{K^*}} \cdot J^2 K^{12} (J^2 \epsilon_{K^*}^2 \mu^{2_{K^*} \times 1}) &= \\ &= J^2 (J^2 K^{12} \epsilon_{K^*}^2) \mu^{2_{K^*} \times 1} = J^2 K^{12} \mu^{2_{K^*} \times 1}. \end{aligned}$$

This proves that $J^2 \epsilon_{K^*}^2 \mathcal{M}^{2_{K^*} \times 1} \subset \mathcal{M}^{2_{K^*} \times 1}$. On the other hand, by 22.31), we have $\mathcal{M}^{2_{K^*} \times 1} \supset \mathcal{M}^{2_{K^*} \times 1}$, and hence $J^2 \epsilon_{K^*}^2 \mathcal{M}^{2_{K^*} \times 1} \supset J^2 \epsilon_{K^*}^2 \mathcal{M}^{2_{K^*} \times 1} = \mathcal{M}^{2_{K^*} \times 1}$. Therefore, $\mathcal{M}^{2_{K^*} \times 1} = J^2 \epsilon_{K^*}^2 \mathcal{M}^{2_{K^*} \times 1}$. 22.33) is obtained from 22.32) by putting ϵ_K^1 in place of ϵ_K^1 . A similar remark applies to 22.42).

By IV A, T V), we note that

$$\mu^{2_{K^*} \times 1} \cdot \mathbb{E} | (\mu^{2_{K^*}} \cdot \mu_o^{O^2 \mathcal{M}^{2_{K^*}}}) \ni \mu^{2_{K^*} \times 1} = \mu^{2_{K^*}} + \mu_o^2.$$

It suffices to show that this $\mu^{2_{K^*}}$ is in $\mathcal{M}^{2_{K^*} \times 1}$. To prove this, we note that $J^2 K^{12} \mu_o^2 = O^1$ since K^{12} rows $M^{2_{K^*}}$. Therefore

$$J^2 K^{12} \mu^{2_{K^*} \times 1} = J^2 K^{12} (\mu^{2_{K^*}} + \mu_o^2) = J^2 K^{12} \mu^{2_{K^*}},$$

from which we get $J^2 K^{12} \mu^{2_{K^*}}$ is M^1 .

In the statement 22.51), we see that $\mathcal{M}^{1_{K^*}} \supset \mathcal{M}_K^{1 \cdot 2}$ since $\mathcal{M}^{1_{K^*}} = J^2 K^{12} \mathcal{M}^2$ and $\mathcal{M}^{2_{K^*} \times 1} \subset \mathcal{M}^2$. Next, we shall show that

$$\mathcal{M}_K^{1 \cdot 2_{K^*}} \subset \mathcal{M}_K^{1 \cdot 2} \subset \mathcal{M}_{n(1 \cdot 1_{K^*})}^1 \subset \mathcal{M}_K^{1 \cdot 2_{K^*}}.$$

Now $\mathcal{M}_K^{1 \cdot 2_{K^*}} \subset \mathcal{M}_K^{1 \cdot 2}$ is obvious. From $\mathcal{M}_K^{1 \cdot 2} \subset \mathcal{M}^1$ and $\mathcal{M}_K^{1 \cdot 2} \subset \mathcal{M}^{1_{K^*}}$, we secure $\mathcal{M}_K^{1 \cdot 2} \subset \cap (\mathcal{M}^1 \cdot \mathcal{M}^{1_{K^*}}) \equiv \mathcal{M}_{n(1 \cdot 1_{K^*})}^1$. Since $\mathcal{M}_K^{1 \cdot 2_{K^*}} = J^2 K^{12} \mathcal{M}^{2_{K^*} \times 1}$, the last inclusion will be proved if we can show that

$$\mu_{n(1 \cdot 1_{K^*})}^1 \cdot \mathbb{E} | \mu^{2_{K^*} \times 1} \ni \mu_{n(1 \cdot 1_{K^*})}^1 = J^2 K^{12} \mu^{2_{K^*} \times 1}.$$

Take any $\mu_{\alpha(\cdot, \cdot, \kappa^*)}^1$ and consider $J^{1\kappa^*} K^{*21} \mu_{\alpha(\cdot, \cdot, \kappa^*)}^1$. Then by IV A, T IV, $J^{1\kappa^*} K^{*21} \mu_{\alpha(\cdot, \cdot, \kappa^*)}^1$ is $M^{2\kappa^*}$, and furthermore,

$$\begin{aligned} J^{2K^{12}}(J^{1\kappa^*} K^{*21} \mu_{\alpha(\cdot, \cdot, \kappa^*)}^1) &= J^{1\kappa^*} (J^{2K^{12}} K^{*21}) \mu_{\alpha(\cdot, \cdot, \kappa^*)}^1 \\ &= J^{1\kappa^*} \varepsilon_{\kappa^*}^1 \mu_{\alpha(\cdot, \cdot, \kappa^*)}^1 = \mu_{\alpha(\cdot, \cdot, \kappa^*)}^{1M^1} \end{aligned}$$

Therefore, define $\mu^{2\kappa^* \kappa^1} \equiv J^{1\kappa^*} K^{*21} \mu_{\alpha(\cdot, \cdot, \kappa^*)}^1$, and the statement is proved.

In the preceding demonstration, with ε^1 replaced by ε_{κ}^1 , we obtain

$$\mathcal{M}_{\kappa}^{1\kappa \cdot 2\kappa^*} \subset \mathcal{M}_{\kappa}^{1\kappa \cdot 2} \subset \mathcal{M}_{\alpha(\cdot, \cdot, \kappa^*)}^1 \subset \mathcal{M}_{\kappa}^{1\kappa \cdot 2\kappa^*}.$$

Finally, that $\mathcal{M}_{\alpha(\cdot, \cdot, \kappa^*)}^1 \supset \mathcal{M}_{\alpha(\cdot, \cdot, \kappa^*)}^1$ is evident. Thus 22.51) is completely proved.

From 22.51) we obtain $J^{2\kappa^*} K^{12} \mathcal{M}^{2\kappa^* \kappa^1} = \mathcal{M}_{\alpha(\cdot, \cdot, \kappa^*)}^1$. Hence

$$\begin{aligned} J^{1\kappa^*} K^{*21} \mathcal{M}_{\alpha(\cdot, \cdot, \kappa^*)}^1 &= J^{1\kappa^*} K^{*21} (J^{2\kappa^*} K^{12} \mathcal{M}^{2\kappa^* \kappa^1}) \\ &= J^{2\kappa^*} (J^{1\kappa^*} K^{*21} K^{12}) \mathcal{M}^{2\kappa^* \kappa^1} = J^{2\kappa^*} \varepsilon_{\kappa^*}^2 \mathcal{M}^{2\kappa^* \kappa^1} = \mathcal{M}^{2\kappa^* \kappa^1}. \end{aligned}$$

For 22.53), we note that K^{12} comp $\overline{\text{rows}} \mathcal{M}^2$ gives $\varepsilon^2 = \varepsilon_{\kappa^*}^2$, and hence by 22.52), we have the necessary condition. Conversely, if the condition is satisfied, then every μ^2 for which $J^{2\overline{\mu^2}} K^{*21} = 0^1$ is in $\mathcal{M}^{2\kappa^*}$, and hence is 0^2 , since K^{12} comp $\overline{\text{rows}} \mathcal{M}^{2\kappa^*}$.

22.54) is secured from 22.52) when ε^1 is replaced by ε_{κ}^1 .

22.55) follows from 22.54) and 22.51).

To prove 22.56), we shall adopt the following scheme:

(2) $\rightarrow$ (1) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (6) $\rightarrow$ (7) $\rightarrow$ (2), (5) $\sim$ (3).

(2) $\rightarrow$ (1) is evident. That (1) implies (3) follows from 22.51).

To show (3) implies (4), it suffices to prove $\mathcal{M}^{2\kappa^1} \subset \mathcal{M}^{2\kappa^1 \kappa}$, since by 22.31) $\mathcal{M}^{2\kappa^1 \kappa} \subset \mathcal{M}^{2\kappa^1}$. By D 22) and hypothesis, we have

$$J^{2K^{12}} \mathcal{M}^{2\kappa^1} = \mathcal{M}_{\kappa}^{1 \cdot 2} = \mathcal{M}_{\kappa}^{1\kappa \cdot 2} \subset \mathcal{M}^{1\kappa}.$$

Thus $\mu^{2 \times 1}$ is transformable by $J^2 K^{12}$ into $\mu^{1 \times}$; in other words, $\mathcal{M}^{2 \times 1} \subset \mathcal{M}^{2 \times 1 \times}$. To prove (4) implies (6), we note by hypothesis that $J^2 K^{12} \mu^{2 \times 1}$ is $M^{1 \times}$ for every $\mu^{2 \times 1}$. Hence by IV A, T II, we secure

$$J^1 \overline{\varphi^{1 \times}} J^2 K^{12} \mu^{2 \times 1} = J^{1 \times} \overline{\varphi^{1 \times}} J^2 K^{12} \mu^{2 \times 1} \quad (\varphi^{1 \times} F^{1 \times}, \mu^{2 \times 1}).$$

Now K^{12} is by cols $M^{1 \times}$ and hence is also by cols $A^{1 \times}$. It follows from III A, T II,

$$J^{1 \times} \overline{\varphi^{1 \times}} J^2 K^{12} \mu^{2 \times 1} = J^2 (J^{1 \times} \overline{\varphi^{1 \times}} K^{12}) \mu^{2 \times 1} \quad (\varphi^{1 \times} F^{1 \times}, \mu^{2 \times 1}).$$

Again, by IV A, T II, the expression on the right can be written as $J^2 (J^1 \overline{\varphi^{1 \times}} K^{12}) \mu^{2 \times 1}$. Thus (6) is proved. That (6) implies (7) follows from 22.31). To prove (7) implies (2), let $\varphi^{1 \times} \equiv \varepsilon_{\kappa}^1(p^1)$. Then

$$J^1 \varepsilon_{\kappa}^1 J^2 K^{12} \mu^{2 \times \times 1} = J^2 (J^1 \varepsilon_{\kappa}^1 K^{12}) \mu^{2 \times \times 1} = J^2 K^{12} \mu^{2 \times \times 1} \quad (\mu^{2 \times \times 1}),$$

which by IV A, T IV implies that $J^2 K^{12} \mu^{2 \times \times 1}$ is $M^{1 \times}$ for every $\mu^{2 \times \times 1}$. Thus by D 22) and 22.51), (2) is established. Finally, from the equivalence of (3) and (4), we secure as a special case,

$$\mathcal{M}_{\kappa}^{1 \cdot 2 \times \times} = \mathcal{M}_{\kappa}^{1 \times \cdot 2 \times \times} \sim \mathcal{M}^{2 \times \times \times 1} = \mathcal{M}^{2 \times \times \times 1 \times}.$$

But by 22.51)

$$\mathcal{M}_{\kappa}^{1 \cdot 2 \times \times} = \mathcal{M}_{\kappa}^{1 \times \cdot 2 \times \times} \sim \mathcal{M}_{\kappa}^{1 \cdot 2} = \mathcal{M}_{\kappa}^{1 \times \cdot 2}.$$

Hence we have the equivalence of (5) and (3).

The following theorem can be proved:

$$(1) \quad \varepsilon_{\kappa}^1 \text{ type } F^1 \overline{F^1} \quad :): (2) \quad J^{1 \times} = J^1 \text{ as on } \overline{\mathfrak{F}^{1 \times}} \mathcal{A}^{1 \times} :):$$

$$(3) \quad \mu^1 A^{1 \times} \cdot) \cdot \mu^1 M^{1 \times} \quad :): (4) \quad \mathcal{M}_{\cap(1 \times 1 \times)}^1 = \mathcal{M}_{\cap(1 \times 1 \times)}^1.$$

$$D\ 24) \ K^{12} T_S^{12} \equiv K^{12} T^{12} \cdot \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 = \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 \cdot \mathcal{M}_{\cap(2_{\kappa})}^2 = \mathcal{M}_{\cap(2_{\kappa} \ast 2_{\kappa})}^2$$

A matrix with property T_S^{12} is indispensable in the study of classical reciprocals.

We should note that, in general, $J^2 K^{12} \mathcal{M}_{\cap(2_{\kappa} \ast 2_{\kappa})}^2 \subset \mathcal{M}^{1\kappa}$. For, if $J^2 K^{12} \mathcal{M}_{\cap(2_{\kappa} \ast 2_{\kappa})}^2 \subset \mathcal{M}^{1\kappa}$, then, since K^{12} is by $\overline{\text{rows}} \mathcal{M}_{\cap(2_{\kappa} \ast 2_{\kappa})}^2$,

$$\varepsilon_{\kappa}^1(\cdot p^1) = J^2 K^{12} K^{\ast 21}(\cdot p^1) M^{1\kappa}(p^1),$$

which is not always true for every $K^{12} T^{12}$, as the following example shows:

$$\mathcal{U}^D = \mathcal{R} \cdot \mathcal{P}' = \mathcal{P}^2 = \mathcal{P}^{\text{III}} \cdot \quad \varepsilon^1 = \varepsilon^2 = \delta.$$

$$K^{12} \equiv K \equiv \left\| \begin{array}{ccccccc} 1, & 1/2, & 1/3, & \dots & 1/n, & \dots & \\ 1/2, & 2, & 0, & \dots & 0, & \dots & \\ 1/3, & 0, & 3, & \dots & 0, & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 1/n, & 0, & 0, & \dots & n, & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \end{array} \right\|$$

In this case, K^{12} comp cols $\mathcal{M}$ $\cdot$ comp $\overline{\text{rows}} \mathcal{M}$. Hence $\varepsilon_{\kappa}^1 = \varepsilon^1 = \delta$. Take $p_1 = 1$. Then $K(\cdot p_1) = (1, 1/2, \dots)$, and

$$\xi \equiv \sum K K(\cdot p_1) = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}, \frac{1}{2} + 1, \dots, \frac{1}{n} + 1, \dots \right).$$

Hence $\sum K K(\cdot p_1)$ is not modular relative to ε_{κ}^1 .

$$T\ 23) \quad 23.1) \quad O' \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 = O' \mathcal{M}^{1\kappa}$$

$$23.2) \quad O'_{\mathcal{M}^{1\kappa}} \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 = [O']$$

$$23.3) \quad O' O' \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 = \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 \sim K^{12} M^{1\kappa \ast 2\kappa}$$

$$23.4) \quad \mathcal{M}_{\cap('_{\kappa'}'_{\kappa'})}^1 L_J^1 = \mathcal{M}^{1\kappa}$$

$$23.5) \quad O'^{\kappa*} \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} = O'^{\kappa*} \mathcal{M}'^{\kappa*} = [O']$$

$$23.6) \quad \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*}) L J^{\kappa*}} = \mathcal{M}'^{\kappa*}$$

To prove 23.1), we note that $\mathcal{M}'^{\kappa} \supset \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})}$ and hence $O' \mathcal{M}'^{\kappa} \subset O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})}$. But $[\text{cols } K^{12}] \subset \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})}$, and it follows that $O'[\text{cols } K^{12}] \supset O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})}$. By IV A, T V, $O'[\text{cols } K^{12}] \subset O' \mathcal{M}'^{\kappa}$. Therefore, the equality follows.

Now $\mathcal{M}'^{\kappa} \supset \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} L J^1$; hence by IV B, T X,

$$O'_{\mathcal{M}'^{\kappa}} \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} = \cap(\mathcal{M}'^{\kappa}, O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})}) = \cap(\mathcal{M}'^{\kappa}, O' \mathcal{M}'^{\kappa}),$$

the last equality being secured from 23.1). Thus, 23.2) follows.

$$\text{By IV B, T V, } O' O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} = \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} \sim \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} L J^1.$$

The equivalence in 23.3) now follows from 21.6).

From 23.1), we obtain $O' O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} = \mathcal{M}'^{\kappa}$. Hence by IV B,

$$\mathcal{M}'_{\cap('_{\kappa} '_{\kappa*}) L J^1} = O' O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*}) L J^1} = O' O' \mathcal{M}'_{\cap('_{\kappa} '_{\kappa*})} = \mathcal{M}'^{\kappa}.$$

23.5) and 23.6) are instances of 23.1) and 23.4) respectively, using $(\mathcal{U}^D \cdot \mathfrak{P}^1 \cdot \mathfrak{P}^2 \cdot \mathcal{E}'_{\kappa*} \cdot \mathcal{E}_{\kappa}^2 \cdot K^{12 T'^{\kappa*} 2\kappa})$ as basis.

$$T 24) \quad 24.1) \quad \mathcal{M}_{\kappa}^{1\kappa* \cdot 2\kappa} = \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1$$

$$24.2) \quad \mathcal{M}^{2\kappa \times 1\kappa*} = J^{1\kappa} K^{*21} \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1 = J^{1\kappa} K^{*21} \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1$$

$$24.3) \quad \mathcal{M}^{2\kappa \times 1\kappa*} \begin{pmatrix} \mathcal{E}_{\kappa} & \mathcal{K}_{\kappa*} \\ \mathcal{I}_{\kappa} & \mathcal{K}_{\kappa*} \end{pmatrix} \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1$$

$$24.4) \quad \mathcal{M}^{2\kappa \times 1\kappa*} = \mathcal{M}_{\cap(2\kappa* 2\kappa)}^2 \sim \mathcal{M}^{1\kappa \times \kappa* 2\kappa*} = \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1$$

$$\cdot) \cdot \mathcal{E}_{\kappa}^{2 T^{2\kappa*} 2\kappa*} \quad \mathcal{E}_{\kappa}^{1 T^{1\kappa*} 1\kappa*}$$

$$24.5) \quad (1) \quad \mathcal{M}^{2\kappa \times 1\kappa*} = \mathcal{M}^{2\kappa* \times 1\kappa} \sim$$

$$(2) \quad \mathcal{M}^{2\kappa*} \supset \mathcal{M}^{2\kappa \times 1\kappa*} \begin{pmatrix} 2\kappa* & \kappa \\ \mathcal{I}_{\kappa*} & \mathcal{K}_{\kappa*} \end{pmatrix} \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1 \sim$$

$$(3) \quad \mathcal{M}^{2\kappa} \supset \mathcal{M}^{2\kappa* \times 1\kappa} \begin{pmatrix} 2\kappa & \kappa \\ \mathcal{I}_{\kappa} & \mathcal{K}_{\kappa*} \end{pmatrix} \mathcal{M}_{\cap('_{\kappa} '_{\kappa*})}^1 \cdot)$$

$$(4) \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})} \subset \Omega(\mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K} \cdot \mathcal{M}^{\mathbf{1}_{K^*} \mathbf{K}^* \mathbf{2}_K})$$

$$24.6)^1 (1) \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}} = \mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K} = \mathcal{M}^{\mathbf{1}_{K^*} \mathbf{K}^* \mathbf{2}_K} \sim.$$

$$(2) \mathcal{M}_{\Omega(\mathbf{2}_K \mathbf{2}_{K^*})}^{\mathbf{2}} = \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_K} = \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_{K^*}} \sim.$$

$$(3) \mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K} = \mathcal{M}^{\mathbf{1}_{K^*} \mathbf{K}^* \mathbf{2}_K} \cdot \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_K} = \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_{K^*}}$$

$$24.7) K^{\mathbf{12}} M^{\mathbf{1}_K \mathbf{2}_K} \sim. \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_{K^*}} = \mathcal{M}^{\mathbf{2}_K}$$

$$24.81) K^{\mathbf{12}} M^{\mathbf{12}} :) : K^{\mathbf{12}} M^{\mathbf{1}_K \mathbf{2}_K} \sim. \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}} = \mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K}$$

$$24.82) K^{\mathbf{12}} M^{\mathbf{1}_K \mathbf{2}_K} :) : K^{\mathbf{12}} M^{\mathbf{12}} \sim. \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}} = \mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K}$$

From IV A theory, $K^{\mathbf{12}} T^{\mathbf{1}_K \mathbf{2}_K}$ comp cols $\mathcal{M}^{\mathbf{1}_K}$ comp rows $\overline{\mathcal{M}^{\mathbf{2}_K}}$ and hence $(\varepsilon_{K^*}^{\mathbf{1}} \cdot \varepsilon_K^{\mathbf{2}} \cdot \varepsilon_{K^*}^{\mathbf{1}} \cdot \varepsilon_K^{\mathbf{2}} \cdot \varepsilon_K^{\mathbf{1}} \cdot \varepsilon_{K^*}^{\mathbf{2}})$ is an instance of $(\varepsilon^{\mathbf{1}} \cdot \varepsilon^{\mathbf{2}} \cdot \varepsilon_K^{\mathbf{1}} \cdot \varepsilon_{K^*}^{\mathbf{2}} \cdot \varepsilon_{K^*}^{\mathbf{1}} \cdot \varepsilon_K^{\mathbf{2}})$. Applying this fact to T 22.53), we obtain immediately the result in 24.2). 24.1) follows from 24.2) and the definition D 22). From IV A, $\mathcal{M}^{\mathbf{2}_K} \left(\begin{smallmatrix} \mathbf{2}_K & \mathbf{K} \\ \mathbf{1}_K & \mathbf{K}^* \end{smallmatrix} \right) \mathcal{M}^{\mathbf{1}_K}$, and it is obvious that $\mathcal{M}^{\mathbf{2}_K} \supset \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_{K^*}}$ and $\mathcal{M}^{\mathbf{1}_K} \supset \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}}$. Thus by 24.1) or 24.2), we secure 24.3).

To prove 24.4), we first show that the condition is necessary. From the hypothesis, it follows that

$$J^{\mathbf{2}_K} K^{\mathbf{12}} \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_{K^*}} = J^{\mathbf{2}_K} K^{\mathbf{12}} \mathcal{M}_{\Omega(\mathbf{2}_K \mathbf{2}_{K^*})}^{\mathbf{2}}.$$

But $\mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}} = J^{\mathbf{2}_K} K^{\mathbf{12}} \mathcal{M}^{\mathbf{2}_K \mathbf{K} \mathbf{1}_{K^*}}$ by the definition, D 22) and 24.1), and $J^{\mathbf{2}_K} K^{\mathbf{12}} \mathcal{M}_{\Omega(\mathbf{2}_K \mathbf{2}_{K^*})}^{\mathbf{2}} = \mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K}$ by 22.54), and hence $\mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K} = \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}}$. For the sufficiency, we note that $\mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K} = \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}}$ gives

$$J^{\mathbf{1}_K} K^{\mathbf{21}} \mathcal{M}^{\mathbf{1}_K \mathbf{K}^* \mathbf{2}_K} = J^{\mathbf{1}_K} K^{\mathbf{21}} \mathcal{M}_{\Omega(\mathbf{1}_K \mathbf{1}_{K^*})}^{\mathbf{1}}.$$

¹ We may state two more equivalent conditions in terms of one-to-one correspondences by virtue of 24.5), (2), (3), and their parities.

Applying 24.2) and 22.51) to the preceding relation, we prove the condition is sufficient. For the implication, we have, from the hypothesis, that K^{12} rows $\mathcal{M}^{2_{K^*} \times 1_{K^*}}$, and hence

$$\varepsilon_K^1(.p^1) = J^{2_{K^*}} K^{12} K^{*21}(.p^1) M^{1_{K^*}}(p^1).$$

Similarly, K^{12} cols $\mathcal{M}^{1_{K^*} \times 2_{K^*}}$ and hence

$$\varepsilon_K^2(.p^2) = J^{1_{K^*}} K^{*21} K^{12}(.p^2) M^{2_{K^*}}(p^2).$$

Thus 24.4) is established.

For 24.5), we shall first show (1) implies (2). Now the condition $\mathcal{M}^{2_{K^*}} \supset \mathcal{M}^{2_{K^*} \times 1_{K^*}}$ is evident from (1). To prove the one-to-one correspondence, we observe that $\mathcal{M}_{\cap(1_{K^*} \times 1_{K^*})}^1 \subset \mathcal{M}^{1_{K^*}}$ and $\mathcal{M}^{2_{K^*} \times 1_{K^*}} \subset \mathcal{M}^{2_{K^*}}$, and further from IV A, $\mathcal{M}^{2_{K^*}} ({}^{2_{K^*}}_{1_{K^*}} \times {}^{1_{K^*}}_{K^*}) \mathcal{M}^{1_{K^*}}$. Thus by 22.55) and the hypothesis, we have the one-to-one correspondence in (2). Next, we shall show (2) implies (1). From the one-to-one correspondence, we secure

$$J^{1_{K^*}} K^{*21} \mathcal{M}_{\cap(1_{K^*} \times 1_{K^*})}^1 = \mathcal{M}^{2_{K^*} \times 1_{K^*}}.$$

But the left member equals $\mathcal{M}^{2_{K^*} \times 1_{K^*}}$ by 22.54). Thus (1) follows. We may demonstrate (1)~(3) in a similar fashion. Finally, to prove (4), we secure from (3) that

$$J^{1_{K^*}} K^{*21} \mathcal{M}_{\cap(1_{K^*} \times 1_{K^*})}^1 = \mathcal{M}^{2_{K^*} \times 1_{K^*}},$$

whence by D 21), $\mathcal{M}_{\cap(1_{K^*} \times 1_{K^*})}^1 \subset \mathcal{M}^{1_{K^*} \times 2_{K^*}}$. Similarly, from (2), we secure $\mathcal{M}_{\cap(1_{K^*} \times 1_{K^*})}^1 \subset \mathcal{M}^{1_{K^*} \times 2_{K^*}}$. Thus (4) follows.

That (1)~(2) in 24.6) follows from 24.4) and its hermitian parity. To complete the demonstration, we shall show

$$(1).(2) \rightarrow (3) \rightarrow (1).$$

Now (1).(2) $\rightarrow$ (3) is evident. For (3) $\rightarrow$ (1), we find from (3₂)

and 24.5), (4) that $\mathcal{M}'_{\mathcal{N}(\mathcal{K}^* \mathcal{K})} \subset \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$. From (3₁) and the hermitian parity of 24.5), (3), we secure $\mathcal{M}'_{\mathcal{N}(\mathcal{K}^* \mathcal{K})} \supset \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$, from which (1) follows.

In 24.7), that the condition is necessary follows by the composition theorem of modularity. To prove the sufficiency, we apply the generalized Hellinger-Toeplitz theorem.

To prove 24.81), we find that $K^{12} M^{12}$ gives $\mathcal{M}'_{\mathcal{N}(\mathcal{K}^* \mathcal{K})} = \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$ and $\mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}} = \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$, which gives $\mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}} = \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$. Therefore, by 21.6), K^{12} is $M^{12} E_{\mathcal{K}^*}$. The converse is evident. For 24.82), we find by 21.6) that $\mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}} = \mathcal{M}'_{\mathcal{N}(\mathcal{K}^* \mathcal{K})} = \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$, i. e.

$$J^{12} K^{*21} \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}} \subset \mathcal{M}'_{\mathcal{K}^* \mathcal{K}^* \mathcal{K}}$$

which by the corollary to the generalized Hellinger-Toeplitz theorem implies $K^{12} M^{12} E_{\mathcal{K}^*}$, and hence $K^{12} M^{12}$. The converse is obvious.

§ 5

By 21.3) the four types of modularity that are always present in K^{12} are

$$M^{12} E_{\mathcal{K}}, \quad M^{12} E_{\mathcal{K}^*}, \quad M^{12} E_{\mathcal{K}^* \mathcal{K}}, \quad M^{12} E_{\mathcal{K}^* \mathcal{K}^*}.$$

Hence by III A theory, the interchange of $(J^1, J^{2\mathcal{K}})$, $(J^{1\mathcal{K}}, J^{2\mathcal{K}^*})$, $(J^{1\mathcal{K}^*}, J^2)$, and $(J^{1\mathcal{K}^* \mathcal{K}}, J^{2\mathcal{K}^* \mathcal{K}})$ in the forms containing K^{12} are legitimate for vectors belonging to the appropriate spaces.

We now naturally come to a new question, viz. Is the relation

$$J^2 (J^{1\mathcal{K}^* \mathcal{K}} K^{12}) \mu^{2\mathcal{K}^* \mathcal{K}} = J^{1\mathcal{K}^* \mathcal{K}} J^2 K^{12} \mu^{2\mathcal{K}^* \mathcal{K}} \quad (\mu^{1\mathcal{K}^* \mathcal{K}}, \mu^{2\mathcal{K}^* \mathcal{K}})$$

valid? The answer as expected is negative. For consider

$$\mathcal{R}^D = \mathcal{R} \cdot \mathcal{P}' = \mathcal{P}^2 = \mathcal{P}^{\text{III}} \cdot \mathcal{E}^1 = \mathcal{E}^2 = \mathcal{S}.$$

$$K^{12} = - \sum_n n(n+1) \delta_n \overline{\delta_n} + \sum_n (n+1)^2 \delta_n \overline{\delta_{n+1}}.$$

Take $\mu^{1 \times^* 2} = (-\frac{1}{n+1}|n)$, $\mu^{2 \times 1} = (1/n|n)$. Then $S^1 \mu^{1 \times^* 2} K^{12} = \delta_1^2$, and $S^2 K^{12} \mu^{2 \times 1} = 0^1$, from which it follows readily that

$$1 = S^2 (S^1 \mu^{1 \times^* 2} K^{12}) \mu^{2 \times 1} \neq S^1 \mu^{1 \times^* 2} (S^2 K^{12} \mu^{2 \times 1}) = 0.$$

From D 21), we have eight spaces, namely,

$$\begin{aligned} & \mathcal{M}^{2 \times 1} \cdot \mathcal{M}^{2 \times^* \times 1} \cdot \mathcal{M}^{2 \times 1 \times} \cdot \mathcal{M}^{2 \times^* \times 1 \times} \\ & \cdot \mathcal{M}^{1 \times^* 2} \cdot \mathcal{M}^{1 \times \times^* 2} \cdot \mathcal{M}^{1 \times^* 2 \times^*} \cdot \mathcal{M}^{1 \times \times^* 2 \times^*}, \end{aligned}$$

from which we secure sixteen possible types of interchange of the integration processes J^1 and J^2 , of which the above example is only one. However, it will be shown that these sixteen types of interchange of (J^1, J^2) are not all distinct, and, in fact, can be classified into four equal groups so that all four types in each group are equivalent to each other. If we indicate each type by the two types of vectors that enter it then the four groups are as follows:

$$1.1) (\mu^{1 \times^* 2}, \mu^{2 \times 1}),$$

$$1.2) (\mu^{1 \times \times^* 2}, \mu^{2 \times 1}),$$

$$1.3) (\mu^{1 \times^* 2}, \mu^{2 \times^* \times 1}),$$

$$1.4) (\mu^{1 \times \times^* 2}, \mu^{2 \times^* \times 1});$$

$$2.1) (\mu^{1 \times^* 2 \times^*}, \mu^{2 \times 1}), \quad 3.1) (\mu^{1 \times^* 2}, \mu^{2 \times 1 \times}),$$

$$2.2) (\mu^{1 \times \times^* 2 \times^*}, \mu^{2 \times 1}), \quad 3.2) (\mu^{1 \times^* 2}, \mu^{2 \times^* \times 1 \times}),$$

$$2.3) (\mu^{1 \times^* 2 \times^*}, \mu^{2 \times^* \times 1}), \quad 3.3) (\mu^{1 \times \times^* 2}, \mu^{2 \times 1 \times}),$$

$$2.4) \quad (\mu^{1\kappa\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1}); \quad 3.4) \quad (\mu^{1\kappa\kappa^*2}, \mu^{2\kappa^*\kappa^*1\kappa});$$

$$4.1) \quad (\mu^{1\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1\kappa}),$$

$$4.2) \quad (\mu^{1\kappa\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1\kappa}),$$

$$4.3) \quad (\mu^{1\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1\kappa}),$$

$$4.4) \quad (\mu^{1\kappa\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1\kappa}).$$

We note that 1) and 4) are their own hermitian parities while 2) and 3) are the hermitian parities of each other. We shall prove the equivalence of the four types in each group.

From the spaces $\mathcal{M}^{1\kappa^*\kappa^*2\kappa}$, $\mathcal{M}^{2\kappa\kappa^*1\kappa^*}$, we obtain a new type of interchange of integration processes, designated by $I^{1\kappa^*2\kappa}$ (see D 25). However, the $I^{1\kappa^*2\kappa}$ -property is equivalent to the $I^{1\kappa^*2\kappa^*}$ -property (T 25.3). It is evident that, when K^{12} is complete by cols $\mathcal{M}^1$ and also complete by rows $\mathcal{M}^2$, then all seventeen types of interchange of integration processes are equivalent. They are also all equivalent when K^{12} is of T_s^{12} (T 25.43).

For convenience, we shall adopt the following notation:

$$D 25) \quad K^{12} I^{12} \equiv J^1 \overline{\mu^{1\kappa^*2}} J^2 K^{12} \mu^{2\kappa^*1} = J^2 (J^1 \overline{\mu^{1\kappa^*2}} K^{12}) \mu^{2\kappa^*1} \\ (\mu^{1\kappa^*2}, \mu^{2\kappa^*1}).$$

As special instances of D 25), we have the following additional types: $I^{12\kappa^*}$, $I^{1\kappa^*2}$, $I^{1\kappa^*2\kappa^*}$, $I^{1\kappa^*2\kappa}$, the pairs of vectors that characterize the forms containing K^{12} being $(\mu^{1\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1})$, $(\mu^{1\kappa\kappa^*2}, \mu^{2\kappa^*\kappa^*1\kappa})$, $(\mu^{1\kappa\kappa^*2\kappa^*}, \mu^{2\kappa^*\kappa^*1\kappa})$, $(\mu^{1\kappa^*\kappa^*2\kappa}, \mu^{2\kappa^*\kappa^*1\kappa^*})$ respectively. By IV A theory, we may replace the integration operators $J^{1\kappa}$, $J^{2\kappa^*}$ by J^1 , J^2 respectively in the first three types, $I^{12\kappa^*}$, $I^{1\kappa^*2}$, and $I^{1\kappa^*2\kappa^*}$. With this view, we state the following theorem:

$$T \ 25 \ 25.1) \ J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} = J^2 (J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}}) \mu^{2\kappa^* \kappa^* 1} \\ (\mu^{1\kappa^* \kappa^* 2}, \mu^{2\kappa^* \kappa^* 1})$$

$$\cdot) \cdot \mathcal{M}'_{n(1' \kappa^*)} = \mathcal{M}'_{n(1' \kappa^*)}$$

$$25.21) \ (1) \ K^{12} \ I^{12} \ \sim. \ (2) \ J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} = \\ = J^2 (J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}}) \mu^{2\kappa^* \kappa^* 1} \quad (\mu^{1\kappa^* \kappa^* 2}, \mu^{2\kappa^* \kappa^* 1})$$

$$\sim. \ (3) \ J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} \\ = J^2 (J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}}) \mu^{2\kappa^* \kappa^* 1} \quad (\mu^{1\kappa^* \kappa^* 2}, \mu^{2\kappa^* \kappa^* 1})$$

$$\sim. \ (4) \ J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} \\ = J^2 (J^1 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}}) \mu^{2\kappa^* \kappa^* 1} \quad (\mu^{1\kappa^* \kappa^* 2}, \mu^{2\kappa^* \kappa^* 1})$$

25.22) The four types of interchange of the integration processes, J^1 , J^2 , in each of the groups 2), 3), and 4) are equivalent.

Thus we may state an equivalent form for theorem

$$25.1) \text{ as follows: } K^{12} \ I^{12\kappa^*} \cdot) \cdot \mathcal{M}'_{n(1' \kappa^*)} = \mathcal{M}'_{n(1' \kappa^*)}$$

$$25.3) \ (1) \ K^{12} \ I^{1\kappa^* 2\kappa^*} \ \sim. \ (2) \ K^{12} \ I^{1\kappa^* 2\kappa^*} \ \sim.$$

$$(3) \ J^1 J^2 \mu^{1\kappa^*} \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} = J^1 \mu^{1\kappa^*} J^2 \mu^{2\kappa^*} \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} \\ (\mu^{1\kappa^* \kappa^* 2}, \mu^{2\kappa^* \kappa^* 1})$$

$$25.41) \ K^{12} \ I^{12} \ \cdot) \cdot K^{12} \ I^{12\kappa^*} \ \cdot) \cdot K^{12} \ I^{1\kappa^* 2\kappa^*}$$

$$25.42) \ (1) \ K^{12} \ I^{12} \ \sim. \ (2) \ K^{12} \ I^{12\kappa^*} \cdot I^{1\kappa^* 2\kappa^*} \ \sim.$$

$$\sim. \ (3) \ K^{12} \ I^{1\kappa^* 2\kappa^*} \cdot \mathcal{M}'_{n(1' \kappa^*)} = \mathcal{M}'_{n(1' \kappa^*)} \cdot \mathcal{M}^2_{n(2 \kappa^*)} = \mathcal{M}^2_{n(2 \kappa^* 2 \kappa^*)}$$

$$\sim. \ (4) \ J^1 J^2 \mu^{1\kappa^*} \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} = J^1 \mu^{1\kappa^*} J^2 \overline{\mu^{1\kappa^*} \mu^{2\kappa^*} K^{12}} \mu^{2\kappa^* \kappa^* 1} \\ (\mu^{1\kappa^* \kappa^* 2}, \mu^{2\kappa^* \kappa^* 1})$$

$$25.43) K^{12} T_s^{12} : (1) K^{12} I^{12} \sim (2) K^{12} I^{12_{K^*}} \sim$$

$$(3) K^{12} I^{1_{K^*} 2} \sim (4) K^{12} I^{1_{K^*} 2_{K^*}}$$

$$25.51) K^{12} I^{12} \sim: \mu^{1_{K^*} 2} \cdot \mu^{2 \times 1} \cdot).$$

$$\sum_{\sigma^2} J^1 \overline{\mu^{1_{K^*} 2}} J_{\sigma^2}^2 K^{12} \mu^{2 \times 1} = J^1 \overline{\mu^{1_{K^*} 2}} J^2 K^{12} \mu^{2 \times 1}$$

$$25.52) \sum_{\sigma^2} J_{\sigma^2}^2 K^{12} \mu^{2 \times 1} = J^2 K^{12} \mu^{2 \times 1} (\mu^{2 \times 1}) \cdot). K^{12} I^{12}$$

The last two statements give us two criteria for testing the I^{12} -property of K^{12} .

By III A, T II, we find that

$$J^{1_{K^*}} K^{21} \varphi^{1_{K^*}} M^{2_{K^*}} (\varphi^{1_{K^*}} F^{1_{K^*}}),$$

from which it follows that $[\text{all } \varphi^{1_{K^*}} F^{1_{K^*}}] \subset \mathcal{M}^{1_{K^*} K^* 2_{K^*}}$. Thus by 22.56), (1) $\sim$ (7), we obtain immediately 25.1).

For the proof of 25.21), we shall adopt the following scheme: (1) $\rightarrow$ (2) $\rightarrow$ (4) $\rightarrow$ (1), from which we secure the rest by parity. Now that (1) $\rightarrow$ (2) $\rightarrow$ (4) is obvious, since $\mathcal{M}^{1_{K^*} K^* 2} \subset \mathcal{M}^{1_{K^*} K^* 2}$ and $\mathcal{M}^{2_{K^*} K^* 1} \subset \mathcal{M}^{2 \times 1}$. It remains to show that (4) implies (1). Consider any $\mu^{1_{K^*} 2}, \mu^{2 \times 1}$. By 22.41),

$$\exists |(\mu^{1_{K^*} 2}, \mu_0^1 O' \mathcal{M}^{1_{K^*}}) \ni \mu^{1_{K^*} 2} = \mu^{1_{K^*} K^* 2} + \mu_0^1,$$

$$\exists |(\mu^{2_{K^*} K^* 1}, \mu_0^2 O^2 \mathcal{M}^{2_{K^*}}) \ni \mu^{2 \times 1} = \mu^{2_{K^*} K^* 1} + \mu_0^2.$$

Therefore

$$\begin{aligned} J^1 \overline{\mu^{1_{K^*} 2}} J^2 K^{12} \mu^{2 \times 1} &= J^1 (\overline{\mu^{1_{K^*} K^* 2} + \mu_0^1}) J^2 K^{12} (\mu^{2_{K^*} K^* 1} + \mu_0^2) \\ &= J^1 \overline{\mu^{1_{K^*} K^* 2}} J^2 K^{12} \mu^{2_{K^*} K^* 1} + J^1 \overline{\mu_0^1} J^2 K^{12} \mu^{2_{K^*} K^* 1} \\ &\quad + J^1 \overline{\mu^{1_{K^*} K^* 2}} J^2 K^{12} \mu_0^2 + J^1 \overline{\mu_0^1} J^2 K^{12} \mu_0^2. \end{aligned}$$

Now K^{12} rows $M^{2_{K^*}}$ and hence $J^2 K^{12} \mu_0^2 = 0^1$. Furthermore, if (4) is

true, then the hypothesis in 25.1) is satisfied, and hence by 22.51), $J^2 K^{12} \mu^{2 \times \times 1} M^{1 \times}$. It follows that $J^1 \overline{\mu^{1 \times \times 2}} J^2 K^{12} \mu^{2 \times \times 1} = 0$. Thus the preceding expression becomes

$$J^1 \overline{\mu^{1 \times \times 2}} J^2 K^{12} \mu^{2 \times \times 1} = J^1 \overline{\mu^{1 \times \times 2}} J^2 K^{12} \mu^{2 \times \times 1}.$$

By the same argument, we secure

$$J^2 (J^1 \overline{\mu^{1 \times \times 2}} K^{12}) \mu^{2 \times \times 1} = J^2 (J^1 \overline{\mu^{1 \times \times 2}} K^{12}) \mu^{2 \times \times 1}.$$

From these two equalities and hypothesis (4), we have $K^{12} I^{12}$.

25.22) can be demonstrated in a similar fashion.

For 25.3), we shall prove $(1) \rightarrow (2) \rightarrow (3) \rightarrow (1)$. To show (1) implies (2), it requires to prove that

$$\begin{aligned} K^{12} I^{1 \times 2 \times *}) \cdot J^{2 \times} (J^{1 \times *} \overline{\mu^{1 \times \times 2 \times} K^{12}}) \mu^{2 \times \times 1 \times *} \\ = J^{1 \times *} \overline{\mu^{1 \times \times 2 \times} J^{2 \times} K^{12} \mu^{2 \times \times 1 \times *}} (\mu^{1 \times \times 2 \times}, \mu^{2 \times \times 1 \times *}). \end{aligned}$$

Consider any $\mu^{1 \times \times 2 \times}$. $\mu^{2 \times \times 1 \times *}$. By 24.3) and its hermitian parity

$$\begin{aligned} \mathfrak{A} | \mu_{\cap(2 \times \times 2 \times)}^2 &\ni \mu^{1 \times \times 2 \times} = J^{2 \times *} K^{12} \mu_{\cap(2 \times \times 2 \times)}^2 = J^2 K^{12} \mu_{\cap(2 \times \times 2 \times)}^2 \\ \cdot \mathfrak{A} | \mu_{\cap(1 \times \times 1 \times *)}^1 &\ni \mu^{2 \times \times 1 \times *} = J^{1 \times} K^{*21} \mu_{\cap(1 \times \times 1 \times *)}^1 = J^1 K^{*21} \mu_{\cap(1 \times \times 1 \times *)}^1. \end{aligned}$$

Therefore

$$\begin{aligned} J^{2 \times} (J^{1 \times *} \overline{\mu^{1 \times \times 2 \times} K^{12}}) \mu^{2 \times \times 1 \times *} \\ = J^{2 \times} (J^{1 \times *} (J^2 \overline{\mu_{\cap(2 \times \times 2 \times)}^2} K^{*21}) K^{12}) J^1 K^{*21} \mu_{\cap(1 \times \times 1 \times *)}^1 \\ = J^{2 \times} (J^2 \overline{\mu_{\cap(2 \times \times 2 \times)}^2} J^{1 \times *} K^{*21} K^{12}) J^1 K^{*21} \mu_{\cap(1 \times \times 1 \times *)}^1 \\ = J^{2 \times} \overline{\mu_{\cap(2 \times \times 2 \times)}^2} J^1 K^{*21} \mu_{\cap(1 \times \times 1 \times *)}^1 \\ = J^1 (J^{2 \times} \overline{\mu_{\cap(2 \times \times 2 \times)}^2} K^{*21}) \mu_{\cap(1 \times \times 1 \times *)}^1 \end{aligned} \quad (1)$$

Similarly,

$$J^{1\kappa*} \overline{\mu^{1\kappa* \kappa* 2\kappa}} J^{2\kappa} K^{12} \mu^{2\kappa \kappa 1\kappa*} = J^{2\kappa} \overline{\mu_{\cap(2\kappa* 2\kappa)}^2} J^{1\kappa*} K^{*21} \mu_{\cap(1\kappa' 1\kappa*)}^1 \quad (11)$$

Hence, if (1) and (11) are proved to be equal, the $I^{1\kappa* 2\kappa}$ -property of K^{12} is established. Now by 22.51), $\exists | (\mu^{1\kappa* \kappa* 2\kappa*} \cdot \mu^{2\kappa* \kappa 1\kappa})$ such that

$$\mu_{\cap(2\kappa* 2\kappa)}^2 = J^{1\kappa* 21} \mu^{1\kappa \kappa* 2\kappa} \quad \mu_{\cap(1\kappa' 1\kappa*)}^1 = J^{2\kappa} K^{12} \mu^{2\kappa* \kappa 1\kappa}.$$

Hence we can write (1) in the form

$$\begin{aligned} & J^{1\kappa} (J^{2\kappa} \overline{\mu_{\cap(2\kappa* 2\kappa)}^2} K^{*21}) \mu_{\cap(1\kappa' 1\kappa*)}^1 \\ &= J^{1\kappa} (J^{2\kappa} (J^{1\kappa} \overline{\mu^{1\kappa \kappa* 2\kappa*}} K^{12}) K^{*21}) J^{2\kappa} K^{12} \mu^{2\kappa* \kappa 1\kappa} \\ &= J^{1\kappa} (J^{1\kappa} \overline{\mu^{1\kappa \kappa* 2\kappa*}} J^{2\kappa} K^{12} K^{*21}) J^{2\kappa} K^{12} \mu^{2\kappa* \kappa 1\kappa} \\ &= J^{1\kappa} \overline{\mu^{1\kappa \kappa* 2\kappa*}} J^{2\kappa} K^{12} \mu^{2\kappa* \kappa 1\kappa} \end{aligned} \quad (111)$$

By the same argument, we can express (11) in the form

$$J^{2\kappa} \overline{\mu_{\cap(2\kappa* 2\kappa)}^2} (J^{1\kappa*} K^{*21} \mu_{\cap(1\kappa' 1\kappa*)}^1) = J^{2\kappa} (J^{1\kappa} \overline{\mu^{1\kappa \kappa* 2\kappa*}} K^{12}) \mu^{2\kappa* \kappa 1\kappa} \quad (1v)$$

By hypothesis, $K^{12} \cdot I^{1\kappa 2\kappa*}$, and so (111) and (1v) are equal, which proves the equality of (1) and (11) as required.

For (2) implies (3), consider any pair $\mu_{\cap(1\kappa' 1\kappa*)}^1 \cdot \mu_{\cap(2\kappa* 2\kappa)}^2$. Since $K^{12} M^{1\kappa 2\kappa} \cdot M^{1\kappa* 2\kappa*}$, we find by III A, T II,

$$\begin{aligned} (J^{1\kappa} J^{2\kappa}) \overline{\mu_{\cap(1\kappa' 1\kappa*)}^1} K^{12} \mu_{\cap(2\kappa* 2\kappa)}^2 &= J^{2\kappa} (J^{1\kappa} \overline{\mu_{\cap(1\kappa' 1\kappa*)}^1} K^{12}) \mu_{\cap(2\kappa* 2\kappa)}^2 \\ &= J^{1\kappa} \overline{\mu_{\cap(1\kappa' 1\kappa*)}^1} (J^{2\kappa} K^{12} \mu_{\cap(2\kappa* 2\kappa)}^2) \end{aligned} \quad (v)$$

Similar relation holds for $J^{1\kappa*} J^{2\kappa*}$. Now by 24.3) and its hermitian parity,

$$\exists | (\mu^{1\kappa* \kappa* 2\kappa} \cdot \mu^{2\kappa \kappa 1\kappa*}) \ni \mu_{\cap(2\kappa* 2\kappa)}^2 = J^{1\kappa*} K^{*21} \mu_{\cap(1\kappa' 1\kappa*)}^1, \quad \mu_{\cap(1\kappa' 1\kappa*)}^1 = J^{2\kappa} K^{12} \mu^{2\kappa* \kappa 1\kappa*},$$

and hence we have

$$\begin{aligned}
& J^{1\kappa*} (J^{2\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{2\kappa*2\kappa}} K^{*2\kappa 1}) \mu_{\cap(1\kappa*1\kappa*)}^1 \\
&= J^{1\kappa*} (J^{2\kappa*} J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa}} K^{12} K^{*2\kappa 1}) J^{2\kappa*} K^{12} \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} \\
&= J^{1\kappa*} (J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa}} J^{2\kappa*} K^{12} K^{*2\kappa 1}) J^{2\kappa*} K^{12} \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} \\
&= J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa}} J^{2\kappa*} K^{12} \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} \quad (vi)
\end{aligned}$$

The conjugate of the hermitian parity of (vi) gives

$$J^{2\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{2\kappa*2\kappa}} J^{1\kappa*} K^{*2\kappa 1} \mu_{\cap(1\kappa*1\kappa*)}^1 = J^{2\kappa*} (J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa}} K^{12}) \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} \quad (vii)$$

By hypothesis, (vi) and (vii) are equal; therefore

$$J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa}} J^{2\kappa*} K^{12} \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} = J^{2\kappa*} (J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa}} K^{12}) \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*}$$

which by (v) and the corresponding equalities in $(J^{1\kappa*}, J^{2\kappa*})$ proves (3).

Finally, for $(3) \rightarrow (1)$, let us consider any $\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa*}$, $\mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*}$. Then by 22.54),

$$\exists \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} \ni \mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa*} = J^{2\kappa*} K^{12} \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*},$$

and

$$\exists \mu_{\cap(1\kappa*1\kappa*)}^1 \ni \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} = J^{1\kappa*} K^{*2\kappa 1} \mu_{\cap(1\kappa*1\kappa*)}^1.$$

By substitution, we can verify that

$$J^{2\kappa*} (J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa*}} K^{12}) \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} = (J^{2\kappa*} J^{1\kappa*}) \overline{\mu_{\cap(2\kappa*2\kappa)}^{2\kappa*2\kappa}} K^{*2\kappa 1} \mu_{\cap(1\kappa*1\kappa*)}^1 \quad (viii)$$

$$\cdot J^{1\kappa*} \overline{\mu_{\cap(2\kappa*2\kappa)}^{1\kappa* \kappa* 2\kappa*}} J^{2\kappa*} K^{12} \mu_{\cap(2\kappa*2\kappa)}^{2\kappa* \kappa 1\kappa*} = (J^{2\kappa*} J^{1\kappa*}) \overline{\mu_{\cap(2\kappa*2\kappa)}^{2\kappa*2\kappa}} K^{*2\kappa 1} \mu_{\cap(1\kappa*1\kappa*)}^1 \quad (ix)$$

Now by hypothesis, (viii) and (ix) are equal, which proves (1).

5.41) is evident from the definitions.

To prove 5.42), we shall use the following scheme:

$$(1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (1).$$

The first implication is obvious; the second one follows from

25.22), 22.31), and 25.1). That (3) implies (4) follows from 25.3) and IV A, T II, 6). To prove (4) implies (1), it suffices to show that (4) implies the relation (4) in 25.21). To do this, we make use of 22.52), and then reason as in the demonstration of (3) $\rightarrow$ (1) in 25.3).

For 25.43), we note that (1) $\rightarrow$ (2) $\rightarrow$ (4) is evident. That (4) $\rightarrow$ (1) follows from the definition of T_g^{12} -property of K^{12} , and 25.42), (1) $\sim$ (3). By parity, we secure the balance of the theorem.

In 25.51), we first show that the condition is necessary. Consider any $u^{1 \times 2} \cdot u^{2 \times 1}$. Then

$$\begin{aligned} J^1 \overline{u^{1 \times 2}} J^2 K^{12} u^{2 \times 1} &= J^2 (J^1 \overline{u^{1 \times 2}} K^{12}) u^{2 \times 1} \\ &= \bigcup_{\sigma_2} \bigcup_{\sigma_1} J_{\sigma_2}^2 (J_{\sigma_1}^1 \overline{u^{1 \times 2}} K^{12}) u^{2 \times 1} = \bigcup_{\sigma_2} \bigcup_{\sigma_1} (J_{\sigma_1}^1 \overline{u^{1 \times 2}} J_{\sigma_2}^2 K^{12} u^{2 \times 1}) \\ &= \bigcup_{\sigma_2} J_{\sigma_2}^1 \overline{u^{1 \times 2}} J_{\sigma_2}^2 K^{12} u^{2 \times 1}. \end{aligned}$$

Secondly, we shall prove the condition is sufficient, for

$$\begin{aligned} J^1 \overline{u^{1 \times 2}} J^2 K^{12} u^{2 \times 1} &= \bigcup_{\sigma_2} J_{\sigma_2}^1 \overline{u^{1 \times 2}} J_{\sigma_2}^2 K^{12} u^{2 \times 1} \\ &= \bigcup_{\sigma_2} J_{\sigma_2}^2 (J_{\sigma_2}^1 \overline{u^{1 \times 2}} K^{12}) u^{2 \times 1} = J^2 (J^1 \overline{u^{1 \times 2}} K^{12}) u^{2 \times 1}. \end{aligned}$$

To prove 25.52), we note from II B, T II,

$$\begin{aligned} J^1 \overline{u^{1 \times 2}} J^2 K^{12} u^{2 \times 1} &= J^1 \overline{u^{1 \times 2}} \bigcup_{\sigma_2, 1'} J_{\sigma_2}^2 K^{12} u^{2 \times 1} \\ &= \bigcup_{\sigma_2} J_{\sigma_2}^1 \overline{u^{1 \times 2}} J_{\sigma_2}^2 K^{12} u^{2 \times 1} \quad (u^{1 \times 2}, u^{2 \times 1}) \end{aligned}$$

which, by 25.21), gives $K^{12} I^{12}$.

§ 6

In the balance of the present section, we shall impose some additional conditions on K^{12} . Examples can easily be constructed to show that these conditions do not strengthen K^{12} to be M^{12} nor $M^{1\kappa^* 2\kappa}$. Properties of K^{12} with such additional conditions are studied. An interesting theorem concerning the space $\mathcal{M}^{1\kappa^* 2\kappa^*}$ is established.

$$T\ 26) \quad 26.1) \quad (1) \quad [\overline{\text{rows } K^{12}}] \subset \mathcal{M}^{2\kappa^* 1\kappa^*} \sim. \quad (2) \quad \varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*} \sim.$$

$$(3) \quad \exists \varphi^{21} \text{ cols } M^{2\kappa^*} \ni J^{2\kappa^*} K^{12} \varphi^{21} = \varepsilon_{\kappa}^1 \sim.$$

$$(4) \quad \exists \varphi^{21} \text{ cols } M^2 \ni J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1.$$

$$26.2) \quad \varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*} :): \quad \varphi^{21} = J^{1\kappa^*} K^{*21} \varepsilon_{\kappa}^1 \sim.$$

$$m^{21} \text{ cols } M^{2\kappa^*} \cdot J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1.$$

$$26.3) \quad \varphi^{21} \text{ cols } M^{2\kappa^*} \cdot): \quad \varepsilon_{\varphi}^1 \equiv J^2 \varphi^{*12} \varphi^{21} = J^{2\kappa^*} \varphi^{*12} \varphi^{21}$$

$$\cdot \varepsilon_{\varphi}^2 \equiv J^1 \varphi^{21} \varphi^{*12}.$$

$$26.4) \quad (1) \quad \varphi^{21} \text{ cols } M^{2\kappa^*} \cdot \text{comp cols } \mathcal{M}^{2\kappa^*} \cdot J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1$$

$$\sim (2) \quad \varphi^{21} \text{ cols } M^2 \cdot \mathcal{M}^{2\varphi} \subset \mathcal{M}^{2\kappa^*} \cdot \text{comp cols } \mathcal{M}^{2\kappa^*}$$

$$\cdot J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1.$$

$$\sim (3) \quad \varphi^{21} \text{ cols } M^2 \cdot J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1 \cdot \mathcal{M}^{2\varphi} = \mathcal{M}^{2\kappa^*}$$

$$\cdot (4) \quad \varphi^{21} = J^{1\kappa^*} K^{*21} \varepsilon_{\kappa}^1 \cdot \varphi^{21} \text{ cols } M^{2\kappa^*} \cdot \text{comp cols } \mathcal{M}^{2\kappa^*}$$

$$J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1 \cdot \mathcal{M}^{2\varphi} = \mathcal{M}^{2\kappa^*}$$

$$26.5) \quad \varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*} \cdot \varphi^{21} \text{ cols } M^2 \cdot J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1 \cdot): \quad \varepsilon_{\kappa}^1 M^{1\kappa^* 1\varphi}$$

$$26.6) \quad \varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*} \cdot \varphi^{21} = J^{1\kappa^*} K^{*21} \varepsilon_{\kappa}^1 :):$$

$$\begin{aligned}
 (1) \quad \varepsilon_{\kappa}^1 \text{ comp cols } \mathcal{M}^{1\kappa*} \quad \sim. \quad (2) \quad \varphi^{21} \text{ comp cols } \mathcal{M}^{2\kappa*} \\
 \sim. (3) \quad K^{12} \overline{\text{rows}} M^{2\varphi} \quad \sim. \quad (4) \quad K^{12} = J^{1\varphi} \varepsilon_{\kappa}^1 \varphi^{*12} \\
 26.7) \quad K^{12} M^{12} \cdot \varphi^{21} \text{ cols } M^{22} \cdot J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa}^1 \quad .). \quad \mathcal{M}^{1\kappa} \subset \mathcal{M}^{1\varphi}
 \end{aligned}$$

In fact, the equivalence between (1) and (2) in 26.1) follows from the definition of ε_{κ}^1 . To complete the demonstration, we shall show $(2) \rightarrow (3) \rightarrow (4) \rightarrow (2)$. Now, if $\varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa*}$, then $J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1$ exists. Define $\varphi^{21} \equiv J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1$. Then

$$\varphi^{21} \{ .p^1 \} = J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1 \{ .p^1 \} M^{2\kappa*} (p^1),$$

and also

$$J^2 K^{12} \varphi^{21} = J^2 K^{12} (J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1) = J^{1\kappa*} J^2 K^{12} K^{*21} \varepsilon_{\kappa}^1 = J^{1\kappa*} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1 = \varepsilon_{\kappa}^1.$$

Thus $(2) \rightarrow (3)$ is proved. That $(3) \rightarrow (4)$ is evident. Finally if φ^{21} is by cols M^{22} , then, by virtue of the fact that K^{12} is $M^{1\kappa*2}$ and III A, T II), we secure $\varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa*}$.

In 26.2), that the condition is necessary can be verified readily. To prove the converse, we observe that

$$\begin{aligned}
 J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1 &= J^{1\kappa*} K^{*21} (J^2 K^{12} \varphi^{21}) \\
 &= J^2 (J^{1\kappa*} K^{*21} K^{12}) \varphi^{21} = J^2 \varepsilon_{\kappa}^2 \varphi^{21} = \varphi^{21}.
 \end{aligned}$$

26.3) follows from IVA, T I and T II.

For the demonstration of 26.4), we shall adopt the following scheme: $(1) \rightarrow (2) \rightarrow (3) \rightarrow (1)$. For $(1) \rightarrow (2)$, it suffices to prove only $\mathcal{M}^{2\varphi} \subset \mathcal{M}^{2\kappa*}$. By an application of IV A to φ^{21} with $(\mathcal{U} \cdot \mathcal{P} \cdot \varepsilon_{\kappa}^2)$ as basis, and 26.2), we find that $\varepsilon_{\varphi}^2 M^{2\kappa*2\kappa*}$, and hence by III A, T I, $\mathcal{M}^{2\varphi} \subset \mathcal{M}^{2\kappa*}$. That $(2) \rightarrow (3)$ follows from IV A, T III. For, if φ^{21} is complete by cols $\mathcal{M}^{2\kappa*}$, then $\varepsilon_{\varphi}^2 = \varepsilon_{\kappa}^2$ and

hence $\mathcal{M}^{2\varphi} = \mathcal{M}^{2\kappa^*}$. To show (3) $\rightarrow$ (1), we first note that φ^{21} is by cols $M^{2\varphi}$. Since by hypothesis $\mathcal{M}^{2\varphi} = \mathcal{M}^{2\kappa^*}$, it follows that φ^{21} cols $M^{2\kappa^*}$. Furthermore, φ^{21} comp cols $\mathcal{M}^{2\varphi}$, and hence φ^{21} comp cols $\mathcal{M}^{2\kappa^*}$, since $\varepsilon_{\varphi}^2 = \varepsilon_{\kappa^*}^2$. In (4), it remains to prove $\varphi^{21} = J^{1\kappa^*} K^{21} \varepsilon_{\kappa^*}^1$. From φ^{21} cols $M^{2\kappa^*}$ and $J^2 K^{12} \varphi^{21} = \varepsilon_{\kappa^*}^1$ we find that $\varepsilon_{\kappa^*}^1$ cols $M^{1\kappa^*}$. Thus by 26.2), we secure the result.

$\varepsilon_{\kappa^*}^1 M^{1\kappa^* 1\varphi}$ follows from the fact that $K^{12} M^{1\kappa^* 2}$ and $\varphi^{21} M^{21\varphi}$.

To prove 26.6), we shall show (1) $\rightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (1), and (3) $\rightarrow$ (4) $\rightarrow$ (3). By 26.2), φ^{21} cols $M^{2\kappa^*}$. Consider $J^{2\kappa^*} \overline{\mu^{2\kappa^*}} \varphi^{21} = O^1$. Then $J^{2\kappa^*} \overline{\mu^{2\kappa^*}} J^{1\kappa^*} K^{21} \varepsilon_{\kappa^*}^1 = J^{1\kappa^*} J^{2\kappa^*} \overline{\mu^{2\kappa^*}} K^{21} \varepsilon_{\kappa^*}^1 = O^1$, which by hypothesis yields $J^{2\kappa^*} \overline{\mu^{2\kappa^*}} K^{21} = O^1$, since $K^{12} M^{1\kappa^* 2\kappa^*}$. This again implies $\mu^{2\kappa^*} = O^2$, since K^{12} comp rows $\mathcal{M}^{2\kappa^*}$. This proves (1) $\rightarrow$ (2). By 26.2) and 26.4), we have $\mathcal{M}^{2\varphi} = \mathcal{M}^{2\kappa^*}$, and hence K^{12} rows $M^{2\varphi}$, which proves (2) $\rightarrow$ (3). To prove (3) $\rightarrow$ (1), consider $J^{1\kappa^*} \overline{\mu^{1\kappa^*}} \varepsilon_{\kappa^*}^1 = O^1$. Then $J^{1\kappa^*} \overline{\mu^{1\kappa^*}} (J^2 K^{12} \varphi^{21}) = J^2 (J^{1\kappa^*} \overline{\mu^{1\kappa^*}} K^{12}) \varphi^{21} = O^1$. Since $\varphi^{21} M^{21\varphi}$, $K^{12} M^{1\kappa^* 2}$, and $\varepsilon_{\varphi}^2 M^{22}$, we have

$$\begin{aligned} O^2 &= J^{1\varphi} (J^2 (J^{1\kappa^*} \overline{\mu^{1\kappa^*}} K^{12}) \varphi^{21}) \varphi^{*12} \\ &= J^2 (J^{1\kappa^*} \overline{\mu^{1\kappa^*}} K^{12}) J^{1\varphi} \varphi^{21} \varphi^{*12} \\ &= J^2 (J^{1\kappa^*} \overline{\mu^{1\kappa^*}} K^{12}) \varepsilon_{\varphi}^2 \\ &= J^{1\kappa^*} \overline{\mu^{1\kappa^*}} J^2 K^{12} \varepsilon_{\varphi}^2 = J^{1\kappa^*} \overline{\mu^{1\kappa^*}} K^{12}. \end{aligned}$$

This implies that $\mu^{1\kappa^*} = O^1$, since K^{12} comp cols $\mathcal{M}^{1\kappa^*}$. Thus (3) implies (1) is proved.

To prove (3) is equivalent to (4), we note that $\varphi^{21} M^{21\varphi}$, and by 26.2) and 26.5), $\varepsilon_{\kappa^*}^1$ rows $M^{1\varphi}$. Thus

$$J^{1\varphi} \varepsilon_{\kappa^*}^1 \varphi^{*12} = J^{1\varphi} (J^2 K^{12} \varphi^{21}) \varphi^{*12} = J^2 K^{12} J^{1\varphi} \varphi^{21} \varphi^{*12} = J^2 K^{12} \varepsilon_{\varphi}^2.$$

Now if (3) holds, then by IVA, T IV, and the equality just proved,

we have (4). Conversely, if (4) holds, then $K^{12} = J^2 K^{12} \epsilon_\varphi^2$.

Since $\epsilon_\varphi^2 \cdots$, we have K^{12} rows $M^{2\varphi}$. This completes the demonstration of 26.6).

To prove 26.7), we note from IV A that $\varphi^{21} M^{21\varphi}$; further, by 21.5), $K^{12} M^{1\kappa^2}$. Thus by composition of modularity, $\epsilon_\kappa^1 = J^2 K^{12} \varphi^{21} M^{1\kappa^1\varphi}$, which gives $\mathcal{M}^{\kappa^1} \subset \mathcal{M}^{1\varphi}$ by 12.2).

Examples show that ϵ_κ^1 is not necessarily by cols of $\mathcal{M}^{\kappa^*}$. When ϵ_κ^1 cols $M^{1\kappa^*}$, we may apply definition D 21) and obtain spaces $\mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*}$. $\mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*}$. It is readily seen from the generalized Hellinger-Toeplitz theorem that in general, $\mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*} \neq \mathcal{M}^{\kappa^1}$, or $\mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*} \neq \mathcal{M}^{\kappa^1}$.

T 27) 27.1) ϵ_κ^1 cols $M^{1\kappa^*}$:): (1) $K^{12} M^{1\kappa^* 2\kappa} \sim$.

$$(2) \mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*} = \mathcal{M}^{\kappa^*} \sim$$

$$(3) \mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*} = \mathcal{M}^{\kappa^1} \cdot)$$

$$(4) \mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*} = \mathcal{M}^{\kappa^*\kappa^*2\kappa}$$

If we assume that ϵ_κ^1 cols $M^{1\kappa^*}$, we may apply the IV A theory to it using the basis $(\mathcal{U} \cdot \mathcal{P}^1 \cdot \epsilon_\kappa^1)$. The positive hermitian matrix $J^{1\kappa^*} \epsilon_\kappa^1 \epsilon_\kappa^1$ which arises at the first step of the application will be denoted by $\epsilon_{0\kappa}^1$ and the corresponding process by $J^{10\kappa}$. We secure now the following theorem.

$$27.2) \quad \epsilon_\kappa^1 \text{ cols } M^{1\kappa^*} \cdot \varphi^{21} \equiv J^{1\kappa^*} K^{21} \epsilon_\kappa^1 :):$$

$$27.21) \quad \epsilon_{0\kappa}^1 = \epsilon_\varphi^1 \cdot M^{1\kappa^* 1\varphi} \epsilon_\kappa^1 = 1 \quad (K^{12} \neq 0^{12}) \cdot \mathcal{M}_{\cap(\kappa^1\varphi)}^{\kappa^1} \supset [\text{cols } \epsilon_\kappa^1]_L$$

$$27.22) \quad \epsilon_\kappa^1 \text{ comp cols } \mathcal{M}^{1\varphi} \cdot \text{comp rows } \mathcal{M}^{1\varphi}$$

$$27.23) \quad \epsilon_\kappa^1 I^{1\kappa^* 1\kappa^*} \cdot) \cdot \mathcal{M}^{\kappa^1\epsilon_\kappa^1\kappa^*} \subset \mathcal{M}^{\kappa^*\kappa^*2\kappa}$$

$$27.24 \quad \mathcal{M}^{1_{\kappa} \epsilon_{\kappa}^{1_{\kappa^*}}} = \mathcal{M}_{\cap(1_{\kappa} 1_{\kappa^*})}^1$$

$$27.25) \quad \mathcal{M}^{1_{\kappa} \epsilon_{\kappa}^{1_{\varphi}}} = \mathcal{M}_{\cap(1_{\kappa} 1_{\varphi})}^1$$

$$27.26) \quad J^{1_{\kappa^*} \epsilon_{\kappa}^{1_{\kappa}}} \mathcal{M}^{1_{\kappa^*} \epsilon_{\kappa}^{1_{\kappa}}} = \mathcal{M}_{\cap(1_{\kappa} 1_{\varphi})}^1$$

$$27.27) \quad \mathcal{M}^{1_{\kappa^*} \epsilon_{\kappa}^{1_{\kappa}}} \subset \mathcal{M}^{1_{\kappa^*} \kappa^{2_{\kappa}}} \cdot \varphi^{21} \overline{\text{rows } M^{1_{\kappa}}} \cdot \epsilon_{\kappa^*}^{21} \overline{\text{rows } M^{2_{\kappa}}}$$

$$27.28) \quad (1) \epsilon_{\kappa}^{11} I^{1_{\kappa} 1_{\kappa^*}} \sim (2) \epsilon_{\kappa}^{11} I^{1_{\kappa^*} 1_{\kappa}} \sim (3) J^{1_{\kappa}} \overline{\mathcal{M}_{\cap(1_{\kappa} 1_{\kappa^*})}^1} \mathcal{M}_{\cap(1_{\kappa} 1_{\varphi})}^1 = \\ = J^2 (J^{1_{\kappa^*}} \overline{\mathcal{M}_{\cap(1_{\kappa} 1_{\kappa^*})}^1} K^{12}) (J^{1_{\varphi}} \varphi^{21} \mathcal{M}_{\cap(1_{\kappa} 1_{\varphi})}^1) \quad (\mathcal{M}_{\cap(1_{\kappa} 1_{\kappa^*})}^1 \cdot \mathcal{M}_{\cap(1_{\kappa} 1_{\varphi})}^1) \\ \sim (4) \epsilon_{\kappa}^{11} I^{1_{\kappa} 1_{\varphi}} \cdot \text{comp cols } \mathcal{M}^{1_{\kappa^*}}$$

$$27.3) \quad \epsilon_{\kappa}^{11} \text{cols } M^{1_{\kappa^*}} \cdot \text{comp cols } \mathcal{M}^{1_{\kappa^*}} \cdot \varphi^{21} \equiv J^{1_{\kappa^*}} K^{21} \epsilon_{\kappa}^{11} \quad (:) :$$

$$27.31) \quad J^{1_{\kappa}} \epsilon_{\kappa}^{11} \epsilon_{\kappa}^{11} = J^{1_{\varphi}} \epsilon_{\kappa}^{11} \epsilon_{\kappa}^{11} = \epsilon_{\kappa^*}^{11}$$

$$27.32) \quad \mathcal{M}^{1_{\kappa^*} \epsilon_{\kappa}^{1_{\kappa}}} = J^{1_{\varphi}} \epsilon_{\kappa}^{11} \mathcal{M}_{\cap(1_{\kappa} 1_{\varphi})}^1$$

$$27.33) \quad \mathcal{M}^{1_{\varphi} \epsilon_{\kappa}^{1_{\kappa}}} = J^{1_{\kappa^*}} \epsilon_{\kappa}^{11} \mathcal{M}_{\cap(1_{\kappa} 1_{\kappa^*})}^1$$

$$27.34) \quad J^{1_{\varphi}} \epsilon_{\kappa}^{11} \mathcal{M}^{1_{\varphi} \epsilon_{\kappa}^{1_{\kappa}}} = \mathcal{M}_{\cap(1_{\kappa} 1_{\kappa^*})}^1$$

To prove 27.1), we first prove the equivalence between (1) and (2). From 21.6), we find that $K^{12} M^{1_{\kappa^*} 2_{\kappa}}$ implies $\epsilon_{\kappa}^{11} M^{1_{\kappa} 1_{\kappa^*}}$. Thus by the composition theorem of modularity, we obtain

$$J^{1_{\kappa^*}} \epsilon_{\kappa}^{11} \mathcal{M}^{1_{\kappa^*} 2_{\kappa}} M^{1_{\kappa}} \quad (\mathcal{M}^{1_{\kappa^*}}).$$

This implies that $\mathcal{M}^{1_{\kappa^*}} \subset \mathcal{M}^{1_{\kappa^*} \epsilon_{\kappa}^{1_{\kappa}}}$, and hence (2) follows. Conversely, if (2) holds, then by the corollary to Hellinger-Toeplitz theorem, ϵ_{κ}^{11} is $M^{1_{\kappa} 1_{\kappa^*}}$ which by 21.6) gives $K^{12} M^{1_{\kappa^*} 2_{\kappa}}$. That (1) is equivalent to (3) can be proved in the same fashion. Finally, (4) is secured from the parity of 24.7) and (2) just proved.

To prove 27.21), we have $J^2 K^{12} \varphi^{21} = \varepsilon_\kappa^1$ by 26.2). Hence

$$\begin{aligned} J^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 &= J^{1\kappa*} (J^2 K^{12} \varphi^{21})^* (J^2 K^{12} \varphi^{21}) = J^{1\kappa*} (J^2 \varphi^{*12} K^{*21}) (J^2 K^{12} \varphi^{21}) \\ &= J^2 \varphi^{*12} (J^{1\kappa*} K^{*21} J^2 K^{12} \varphi^{21}) = J^2 \varphi^{*12} (J^2 J^{1\kappa*} K^{*21} K^{12} \varphi^{21}) \\ &= J^2 \varphi^{*12} J^2 \varepsilon_{\kappa*}^2 \varphi^{21} = J^2 \varphi^{*12} \varphi^{21} = \varepsilon_\varphi^1, \end{aligned}$$

from which the first and third parts follows. For the second part, we use IV A, T II.

That ε_κ^1 comp rows $\mathcal{M}^{1\varphi}$ follows from IV A, T IV. Since ε_κ^1 is hermitian, we have also ε_κ^1 comp cols $\mathcal{M}^{1\varphi}$.

To prove 27.23), consider any $\mu^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1$. Then

$$\begin{aligned} J^{1\kappa*} K^{*21} \mu^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 &= J^{1\kappa*} (J^1 K^{*21} \varepsilon_\kappa^1) \mu^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 \\ &= J^1 K^{*21} (J^{1\kappa*} \varepsilon_\kappa^1 \mu^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1) M^{2\kappa}. \end{aligned}$$

Hence $\mathcal{M}^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 \subset \mathcal{M}^{1\kappa*} \kappa^{*2\kappa}$.

Since $\mu^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 = J^{1\kappa*} \varepsilon_\kappa^1 \mu^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 M^{1\kappa*}$, we have $\mathcal{M}_{\cap(\cdot)' \kappa^{*2\kappa}}^1 \supset \mathcal{M}^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1$. On the other hand, $J^{1\kappa*} \varepsilon_\kappa^1 \mu_{\cap(\cdot)' \kappa^{*2\kappa}}^1 = \mu_{\cap(\cdot)' \kappa^{*2\kappa}}^1$, which is $M^{1\kappa*}$ for every $\mu_{\cap(\cdot)' \kappa^{*2\kappa}}^1$, and hence $\mathcal{M}_{\cap(\cdot)' \kappa^{*2\kappa}}^1 \subset \mathcal{M}^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1$. The equality in 27.24) now follows.

27.25) is an instance of 27.24) by a properly chosen basis. It may also be obtained by reasoning as in 27.24).

Since $\varepsilon_\varphi^1 = J^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1$, 27.26) is an instance of 22.51), which states $\mathcal{M}_{\cap(\cdot)' \kappa^{*2\kappa}}^{1,2} = \mathcal{M}_{\cap(\cdot)' \kappa^{*2\kappa}}^1$, using the basis $(\mathcal{U} \cdot \mathcal{B} \cdot \varepsilon_\kappa^1 \cdot \varepsilon_{\kappa*}^1 \cdot \varepsilon_\kappa^1 \cdot T^{1\kappa' \kappa*})$.

To prove 27.27), we note by hypothesis that $\varphi^{*12} = J^{1\kappa*} \varepsilon_\kappa^1 K^{12}$ which is by cols $M^{1\kappa}$, and hence $[\text{cols } K^{12}] \subset \mathcal{M}^{1\kappa*} \varepsilon_\kappa^1 \varepsilon_\kappa^1 \subset \mathcal{M}^{1\kappa*} \kappa^{*2\kappa}$, whence $J^{1\kappa*} K^{*21} K^{12} = \varepsilon_{\kappa*}^2 \text{cols } M^{2\kappa}$.

In 27.28), the first equivalence is evident. For the second equivalence, consider any pair $\mu_{\cap(\cdot)' \kappa^{*2\kappa}}^1 \cdot \mu_{\cap(\cdot)' \varphi}^1$. Then by 26.2), $J^{1\kappa*} J^{1\varphi} \overline{\mu_{\cap(\cdot)' \kappa^{*2\kappa}}^1} \varepsilon_\kappa^1 \mu_{\cap(\cdot)' \varphi}^1 = J^{1\kappa*} J^{1\varphi} \overline{\mu_{\cap(\cdot)' \kappa^{*2\kappa}}^1} (J^2 K^{12} \varphi^{21}) \mu_{\cap(\cdot)' \varphi}^1 =$

$= J^2(J^{1\kappa*} \overline{p_{n('_{\kappa} '_{\kappa*})}^1} K^{12})(J^{1\varphi} \varphi^{21} p_{n('_{\kappa} '_{\kappa\varphi})}^1)$. By 25.42), (1) $\sim$ (4), using the basis $(\mathcal{U}^D \cdot \mathcal{P}^1 \cdot \varepsilon_{\kappa*}^1 \cdot \varepsilon_{\kappa}^1 \cdot \varepsilon_{\kappa}^1 T^{1\kappa*1\kappa})$ and applying the equality just proved, we obtain (2) $\sim$ (3). To complete the demonstration, we shall show (1).(3) $\rightarrow$ (4) $\rightarrow$ (1). From 25.41), we obtain (1) implies $\varepsilon_{\kappa}^1 T^{1\kappa1\varphi}$. Since K^{12} cols $\mathcal{M}_{n('_{\kappa} '_{\kappa*})}^1$, ε_{κ}^1 cols $\mathcal{M}_{n('_{\kappa} '_{\kappa\varphi})}^1$ and $\varphi^{21} M^{2\kappa*1\varphi}$, we secure from (3), $K^{221} = J^{1\kappa} K^{*21} \varepsilon_{\kappa}^1 = J^2 \varepsilon_{\kappa*}^2 J^{1\varphi} \varphi^{21} \varepsilon_{\kappa}^1 = J^{1\varphi} \varphi^{21} \varepsilon_{\kappa}^1$, which gives ε_{κ}^1 comp cols $\mathcal{M}^{1\kappa*}$, by virtue of 26.6). Finally, that (4) implies (1) is an instances of 25.3), (1) $\sim$ (2), using the basis $(\mathcal{U}^D \cdot \mathcal{P}^1 \cdot \varepsilon_{\kappa*}^1 \cdot \varepsilon_{\kappa}^1 \cdot \varepsilon_{\kappa}^1 T^{1\kappa*1\kappa} \cdot \text{comp cols } \mathcal{M}^{1\kappa*})$.

27.31) follows from 27.21) and IV A, T III.

For 27.32), we note by hypothesis that ε_{κ}^1 comp rows $\mathcal{M}^{1\kappa*}$. Applying 22.53), we have the result.

27.33) is secured by using the same argument as in 27.32).

To prove 27.34), we make use of the results in 27.33), 27.21), and 27.31). Thus

$$\begin{aligned} J^{1\varphi} \varepsilon_{\kappa}^1 \mathcal{M}^{1\varphi} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1 &= J^{1\varphi} \varepsilon_{\kappa}^1 J^{1\kappa*} \varepsilon_{\kappa}^1 \mathcal{M}_{n('_{\kappa} '_{\kappa*})}^1 \\ &= J^{1\kappa*} J^{1\varphi} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1 \mathcal{M}_{n('_{\kappa} '_{\kappa*})}^1 = J^{1\kappa*} \varepsilon_{\kappa*}^1 \mathcal{M}_{n('_{\kappa} '_{\kappa*})}^1 = \mathcal{M}_{n('_{\kappa} '_{\kappa*})}^1. \end{aligned}$$

NOTES. The following theorem can be proved:

A) ε_{κ}^1 cols $M^{1\kappa*}$. $\varphi^{21} \equiv J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1 \cdot \varepsilon_{\kappa}^1 \equiv J^{1\varphi} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1$:);

(1) ε_{κ}^1 comp cols $\mathcal{M}^{1\kappa*}$ $\sim$. (2) K^{12} cols $M^{1\kappa*}$ $\sim$.

(3) $\mathcal{M}_{n('_{\kappa} '_{\kappa*})}^1 = \mathcal{M}_{n('_{\kappa} '_{\kappa*})}$

Denote the modular space relative to $\varepsilon_{\kappa*}^1$ by $\mathcal{M}^{1\kappa*}$, and the corresponding operator by $J^{1\kappa*}$. Let $\varepsilon_{o\kappa*}^1 \sim \mathcal{M}_o^{1\kappa*} \equiv O^{1\kappa*} \mathcal{M}^{1\kappa*}$; then by X B, T X, $\mathcal{M}^{1\kappa*} = \mathcal{M}^{1\kappa*} + \mathcal{M}_o^{1\kappa*}$. We state another theorem.

B) ε_{κ}^1 cols $M^{1\kappa*}$. $\varphi^{21} \equiv J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1 \cdot \varepsilon_{\kappa}^1 \equiv J^{1\varphi} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1$:).

$$\mathcal{M}^{1\kappa*} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1 = J^{1\varphi} \varepsilon_{\kappa}^1 \mathcal{M}_{n('_{\kappa} '_{\kappa\varphi})}^1 + \mathcal{M}_o^{1\kappa*} = J^{1\kappa*} \varepsilon_{\kappa*}^1 \mathcal{M}^{1\kappa*} \varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1 + \mathcal{M}_o^{1\kappa*}$$

$$T\ 28) \quad \varepsilon_K^1 \text{ cols } M^{1_K*} \cdot K^{12} I^{1_K 2_K*} \quad (:) :$$

$$28.1) \quad \mathcal{M}^{1_K* K^* 2_K} \subset \mathcal{M}^{1_K* \varepsilon_K^1 1_K} \cdot J^{1_K*} \varepsilon_K^1 \mathcal{M}^{1_K* K^* 2_K} = \mathcal{M}^{1_K K^* 2_K*}$$

$$28.2) \quad \varphi^{21} \text{ cols } M^2 \cdot J^2 K^{12} \varphi^{21} = \varepsilon_K^1 \cdot .)$$

$$J^2 \varphi^{*12} \mathcal{M}_{\cap(2_K* 2_K)}^2 = \mathcal{M}^{1_K K^* 2_K*} \subset \mathcal{M}_{\cap(1_K 1_\varphi)}^1$$

$$28.3) \quad \varphi^{21} \equiv J^{1_K*} K^{*21} \varepsilon_K^1 \quad (:) :$$

$$28.31) \quad (1) \quad \varepsilon_K^1 \text{ comp cols } \mathcal{M}^{1_K*} \sim. \quad (2) \quad J^{1_\varphi} \varphi^{21} \mathcal{M}_{\cap(1_K 1_\varphi)}^1 \supset \mathcal{M}_{\cap(2_K* 2_K)}^2$$

$$28.32) \quad \varepsilon_K^1 \text{ comp cols } \mathcal{M}^{1_K*} \cdot \mathcal{M}_{\cap(1_K 1_\varphi)}^1 \subset \mathcal{M}^{1_K K^* 2_K*} \quad .)$$

$$(1) \quad J^{1_K} K^{*21} \mu_{\cap(1_K 1_\varphi)}^1 = J^{1_\varphi} \varphi^{21} \mu_{\cap(1_K 1_\varphi)}^1 \quad (\mu_{\cap(1_K 1_\varphi)}^1)$$

$$(2) \quad J^{1_K} K^{*21} \mathcal{M}_{\cap(1_K 1_\varphi)}^1 = J^{1_\varphi} \varphi^{21} \mathcal{M}_{\cap(1_K 1_\varphi)}^1 = \mathcal{M}_{\cap(2_K* 2_K)}^2$$

$$(3) \quad J^{2_K} K^{12} \mu_{\cap(2_K* 2_K)}^2 = J^2 \varphi^{*12} \mu_{\cap(2_K* 2_K)}^2 \quad (\mu_{\cap(2_K* 2_K)}^2)$$

$$(4) \quad J^{2_K} K^{12} \mathcal{M}_{\cap(2_K* 2_K)}^2 = J^2 \varphi^{*12} \mathcal{M}_{\cap(2_K* 2_K)}^2 = \mathcal{M}_{\cap(1_K 1_\varphi)}^1$$

$$28.33) \quad (1) \quad \mathcal{M}^{1_K* \varepsilon_K^1 1_K} \subset \mathcal{M}^{1_K* K^* 2_K}$$

$$\sim. (2) \quad \varepsilon_K^1 \text{ comp cols } \mathcal{M}^{1_K*} \cdot \mathcal{M}_{\cap(1_K 1_\varphi)}^1 \subset \mathcal{M}^{1_K K^* 2_K*}$$

$$\sim. (3) \quad \varphi^{21} \text{ comp cols } \mathcal{M}^{2_K*} \cdot \mathcal{M}_{\cap(1_K 1_\varphi)}^1 \subset \mathcal{M}^{1_K K^* 2_K*}$$

$$\sim. (4) \quad \varepsilon_K^1 \text{ comp cols } \mathcal{M}^{1_K*} \cdot \mathcal{M}_{\cap(2_K* 2_K)}^2 \supset J^{1_\varphi} \varphi^{21} \mathcal{M}_{\cap(1_K 1_\varphi)}^1$$

$$\sim. (5) \quad \mathcal{M}^{1_K* \varepsilon_K^1 1_K} \subset J^2 K^{12} \mathcal{M}_{\cap(2_K* 2_K)}^2$$

$$\sim. (6) \quad \varepsilon_K^1 I^{1_K 1_K*} \cdot .) \cdot (7) \quad J^{1_K*} \varepsilon_K^1 \mathcal{M}^{1_K* \varepsilon_K^1 1_K} = \mathcal{M}^{1_K K^* 2_K*}$$

$$28.34) \quad \varepsilon_{K^*}^2 \text{ cols } M^{2_K} \cdot .) \cdot \varphi^{21} \text{ rows } M^{1_K} \cdot J^{1_\varphi} \varphi^{21} K^{12} = \varepsilon_{K^*}^2 \cdot \varphi^{21} = J^{2_K} \varepsilon_{K^*}^2 K^{*21}$$

For 28.1), it suffices to show the second part, since the first part will then follow as a consequence. From the definition

of ε_{κ}^1 , we secure

$$J^{1\kappa*} \varepsilon_{\kappa}^1 \mathcal{M}^{1\kappa* \kappa^* 2\kappa} = J^{1\kappa*} (J^{2\kappa} K^{12} K^{*21}) \mathcal{M}^{1\kappa* \kappa^* 2\kappa}$$

Since $K^{12} I^{1\kappa 2\kappa*}$ by $\overline{\text{rows}}$ $\mathcal{M}^{2\kappa \kappa^* 1\kappa*}$, we may interchange the integration processes on the right member, by virtue of 25.3). Then by the hermitian parities of 24.1) and 22.54), we secure the required equality.

To prove 28.2), we make use of the hermitian parity of 24.1), and obtain

$$J^{2\varphi*12} \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2 = J^{2\varphi*12} (J^{1\kappa*} K^{*21} \mathcal{M}^{1\kappa* \kappa^* 2\kappa}).$$

After the interchange of integration processes which is legitimate since $K^{12} M^{1\kappa* 2}$, the right member becomes $J^{1\kappa*} \varepsilon_{\kappa}^1 \mathcal{M}^{1\kappa* \kappa^* 2\kappa}$. Thus by 28.1), we secure $J^{2\varphi*12} \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2 = \mathcal{M}^{1\kappa \kappa^* 2\kappa*}$. Now by IV A, $K^{12} M^{1\kappa* 2}$, $\varphi^{21} M^{21\varphi}$, and it follows by the composition theorem of modularity, $\varepsilon_{\kappa}^1 M^{1\varphi 1\kappa*}$. Thus $J^{1\kappa*} \varepsilon_{\kappa}^1 \mathcal{M}^{1\kappa* \kappa^* 2\kappa} \subset \mathcal{M}^{1\varphi}$. Hence $\mathcal{M}^{1\kappa \kappa^* 2\kappa*} \subset \mathcal{M}_{\cap(1\kappa^* 1\varphi)}^1$. This completes the proof of 28.2).

For 28.31), we first prove the condition is necessary. By 28.1), we have $\mathcal{M}^{1\kappa* \kappa^* 2\kappa} \subset \mathcal{M}^{1\kappa*} \varepsilon_{\kappa}^1 1\kappa$. Thus by 27.32) and the hermitian parity of 24.2), we have

$$J^{1\varphi} \varepsilon_{\kappa}^1 \mathcal{M}_{\cap(1\kappa^* 1\varphi)}^1 \supset J^{2} K^{12} \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2$$

from which we secure

$$J^{1\kappa*} K^{*21} J^{1\varphi} \varepsilon_{\kappa}^1 \mathcal{M}_{\cap(1\kappa^* 1\varphi)}^1 \supset J^{1\kappa*} K^{*21} J^{2} K^{12} \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2.$$

Since $\varepsilon_{\kappa}^1 M^{1\kappa* 1\varphi}$ and $K^{12} M^{1\kappa* 2}$, we may interchange the integration processes on both sides of the preceding expression. Then by definitions of φ^{21} and ε_{κ}^2 , the necessary condition is proved.

Since φ^{21} cols M^2 , we note by IV A that $\varphi^{21} M^{2\varphi 1\varphi}$. Hence

by hypothesis, $\mathcal{M}_{\Omega(z_{\kappa^*}, z_{\kappa})}^2 \subset \mathcal{M}^{2\varphi}$. By the hermitian parity of 21.43) we secure $K^{1\kappa}$ rows $\mathcal{M}^{2\varphi}$, which by 26.6), proves the condition is sufficient.

To prove 28.32), we note by 22.51), 26.2), and 28.2) that

$$J^{1\kappa} K^{*\kappa 1} \mathcal{M}_{\Omega('_{\kappa} '_{\varphi})}^1 = J^{1\kappa} K^{*\kappa 1} \mathcal{M}^{1\kappa^* z_{\kappa^*}} = \mathcal{M}_{\Omega(z_{\kappa^*}, z_{\kappa})}^2.$$

With this equality established, if we prove the validity of (1), then (2) will evidently follow. Now consider any $\mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1$. Then $J^{1\varphi} \varphi^{*1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 \overset{M^{2\kappa^*}}{\quad}$ and also by the equality just proved, $J^{1\kappa} K^{*\kappa 1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 \overset{M^{2\kappa^*}}{\quad}$. Thus

$$\begin{aligned} & J^{2\varphi} \varphi^{*12} (J^{1\kappa} K^{*\kappa 1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 - J^{1\varphi} \varphi^{*1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1) \\ &= J^{1\kappa} \mathfrak{e}_{\kappa}^1 \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 - J^{1\varphi} \mathfrak{e}_{\varphi}^1 \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 = \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 - \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 = 0^1. \end{aligned}$$

Now $\mathfrak{e}_{\kappa}^1$ comp cols $\mathcal{M}^{1\kappa^*}$, and hence by 26.6), φ^{*1} comp cols $\mathcal{M}^{2\kappa^*}$.

The result in (1) now follows.

To prove (3) in 28.32), we shall make use of (2). Consider any $\mathfrak{u}_{\Omega(z_{\kappa^*}, z_{\kappa})}^2$. Then

$$\exists \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 \ni J^{1\kappa} K^{*\kappa 1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 = J^{1\varphi} \varphi^{*1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1 = \mathfrak{u}_{\Omega(z_{\kappa^*}, z_{\kappa})}^2 \dots$$

Thus

$$J^{2\kappa} K^{12} \mathfrak{u}_{\Omega(z_{\kappa^*}, z_{\kappa})}^2 = J^{2\kappa} K^{12} (J^{1\kappa} K^{*\kappa 1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1) = \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1,$$

and

$$J^{2\varphi} \varphi^{*12} \mathfrak{u}_{\Omega(z_{\kappa^*}, z_{\kappa})}^2 = J^{2\varphi} \varphi^{*12} (J^{1\varphi} \varphi^{*1} \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1) = \mathfrak{u}_{\Omega('_{\kappa} '_{\varphi})}^1.$$

From these equalities, (3) follows immediately. By 26.2) and 28.2), we secure the last part of 28.32).

In 28.33), we note that (1) $\rightsquigarrow$ (5) follows from the hermitian parity of 24.2), and (2) $\rightsquigarrow$ (3) follows from 26.6). To complete the proof, we shall show (1) $\rightarrow$ (4) $\rightarrow$ (2) $\rightarrow$ (6) $\rightarrow$ (1). For the first implication, we first show

$$\mathcal{M}^{1_{K^*} \epsilon_K^{1_{K^*}}} \subset \mathcal{M}^{1_{K^*} K^{2_{K^*}}} \quad). \quad \epsilon_K^1 \text{ comp cols } \mathcal{M}^{1_{K^*}}$$

Consider $J^{1_{K^*}} \overline{\mu^{1_{K^*}}} \epsilon_K^1 = O^1$. Then $\mu^{1_{K^*}} \subset \mathcal{M}^{1_{K^*} \epsilon_K^{1_{K^*}}}$ and hence by hypothesis, $J^{1_{K^*}} K^{2_{K^*}} \mu^{1_{K^*}} \subset M^{2_{K^*}}$. Thus by 25.3) $O^1 = J^{1_{K^*}} \overline{\mu^{1_{K^*}}} (J^{2_{K^*}} K^{1_{K^*}} K^{2_{K^*}}) = J^{2_{K^*}} (J^{1_{K^*}} \overline{\mu^{1_{K^*}}} K^{1_{K^*}}) K^{2_{K^*}}$, which yields $J^{1_{K^*}} \overline{\mu^{1_{K^*}}} K^{1_{K^*}} = O^2$, since $K^{1_{K^*}}$ is complete by rows $\mathcal{M}^{2_{K^*}}$. This implies $\mu^{1_{K^*}} = O^1$, since $K^{1_{K^*}}$ is complete by cols $\mathcal{M}^{1_{K^*}}$. Therefore, ϵ_K^1 comp cols $\mathcal{M}^{1_{K^*}}$.

By 27.32) and 26.2), we have

$$\mathcal{M}^{1_{K^*} \epsilon_K^{1_{K^*}}} = J^{1_{\varphi}} (J^{2_{K^*}} K^{1_{K^*}} \varphi^{2_{K^*}}) \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1 = J^{2_{K^*}} K^{1_{K^*}} (J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1),$$

and hence $J^{1_{K^*}} K^{2_{K^*}} \mathcal{M}^{1_{K^*} \epsilon_K^{1_{K^*}}} = J^{1_{K^*}} K^{2_{K^*}} (J^{2_{K^*}} K^{1_{K^*}} J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1)$. By 28.1) and the hermitian parity of 24.1), the left member is identical with $\mathcal{M}_{\cap (2_{K^*} 2_{K^*})}^2$. After interchanging J^2 and $J^{1_{K^*}}$ the right member becomes $J^2 (J^{1_{K^*}} K^{2_{K^*}} K^{1_{K^*}}) J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1 = J^2 \epsilon_K^{2_{K^*}} J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1 = J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1$. Thus $\mathcal{M}_{\cap (2_{K^*} 2_{K^*})}^2 = J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1$ and (4) follows.

Since $\varphi^{2_{K^*}} M^{2_{K^*} 1_{\varphi}}$, we have by 28.2)

$$\begin{aligned} \mathcal{M}^{1_{K^*} K^{2_{K^*}}} &= J^2 \varphi^{2_{K^*}} \mathcal{M}_{\cap (2_{K^*} 2_{K^*})}^2 \supset J^2 \varphi^{2_{K^*}} (J^{1_{\varphi}} \varphi^{2_{K^*}} \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1) \\ &= J^{1_{\varphi}} (J^2 \varphi^{2_{K^*}} \varphi^{2_{K^*}}) \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1 = \mathcal{M}_{\cap (1_{K^*} 1_{\varphi})}^1. \end{aligned}$$

To prove (2) $\rightarrow$ (6), take any pair $\mu_{\cap (1_{K^*} 1_{K^*})}^1 \cdot \mu_{\cap (1_{K^*} 1_{\varphi})}^1$. Then $\overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} = J^{1_{K^*}} \overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} J^{2_{K^*}} K^{1_{K^*}} K^{2_{K^*}} = J^2 (J^{1_{K^*}} \overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} K^{1_{K^*}}) K^{2_{K^*}}$. By 22.54) the vector inside the parenthesis is in $\mathcal{M}^{2_{K^*} K^{1_{K^*}}}$. Thus by hypothesis and 28.32), (1), we secure

$$\begin{aligned} J^{1_{K^*}} \overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} \mu_{\cap (1_{K^*} 1_{\varphi})}^1 &= J^{1_{K^*}} (J^2 (J^{1_{K^*}} \overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} K^{1_{K^*}}) K^{2_{K^*}}) \mu_{\cap (1_{K^*} 1_{\varphi})}^1 \\ &= J^2 (J^{1_{K^*}} \overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} K^{1_{K^*}}) (J^{1_{K^*}} K^{2_{K^*}} \mu_{\cap (1_{K^*} 1_{\varphi})}^1) \\ &= J^2 (J^{1_{K^*}} \overline{\mu_{\cap (1_{K^*} 1_{K^*})}^1} K^{1_{K^*}}) (J^{1_{\varphi}} \varphi^{2_{K^*}} \mu_{\cap (1_{K^*} 1_{\varphi})}^1). \end{aligned}$$

Hence by 27.28), we have (6).

For (6) $\rightarrow$ (1), see 27.23). (7) follows from (1) and 28.1).

If $\varepsilon_{\kappa^*}^{\mathfrak{z}}$ is by cols $M^{\mathfrak{z}\kappa}$, then by 26.1) and 25.3)

$$\begin{aligned}\varphi^{*\mathfrak{z}} &= J^{1\kappa^*} \varepsilon_{\kappa^*}^1 K^{1\mathfrak{z}} = J^{1\kappa^*} (J^{\mathfrak{z}\kappa} K^{1\mathfrak{z}} K^{*\mathfrak{z}1}) K^{1\mathfrak{z}} \\ &= J^{\mathfrak{z}\kappa} K^{1\mathfrak{z}} (J^{1\kappa^*} K^{*\mathfrak{z}1} K^{1\mathfrak{z}}) = J^{\mathfrak{z}\kappa} K^{1\mathfrak{z}} \varepsilon_{\kappa^*}^{\mathfrak{z}} \text{ cols } M^{1\kappa}.\end{aligned}$$

It follows that $\varphi^{\mathfrak{z}1} \overline{\text{rows}} M^{1\kappa}$. For the last two parts, we observe by the hermitian parities of 26.1) and 26.2), that

$$\exists \psi^{\mathfrak{z}1} \overline{\text{rows}} M^{1\kappa} \ni J^1 \psi^{\mathfrak{z}1} K^{1\mathfrak{z}} = \varepsilon_{\kappa^*}^{\mathfrak{z}}. \quad \psi^{\mathfrak{z}1} = J^{\mathfrak{z}\kappa} \varepsilon_{\kappa^*}^{\mathfrak{z}} K^{*\mathfrak{z}1}.$$

Thus $\psi^{\mathfrak{z}1} = \varphi^{\mathfrak{z}1}$, and $J^1 \varphi^{\mathfrak{z}1} K^{1\mathfrak{z}} = \varepsilon_{\kappa^*}^{\mathfrak{z}}$ as required.

T 29) $\varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*}. K^{1\mathfrak{z}} M^{1\mathfrak{z}} :): \varepsilon_{\kappa}^1 \text{ comp cols } \mathcal{M}^{1\kappa^*} \leadsto \varepsilon_{\kappa}^1 I^{1\kappa} 1_{\kappa^*}$

$$\leadsto \mathcal{M}^{1\kappa^*} \varepsilon_{\kappa}^1 1_{\kappa} \subset \mathcal{M}^{1\kappa^*} \kappa^{*\mathfrak{z}1}$$

Since $\varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*}$, we may define $\varphi^{\mathfrak{z}1} \equiv J^{1\kappa^*} K^{*\mathfrak{z}1} \varepsilon_{\kappa}^1$.

Then by 22.1), 26.2), and 26.7), we have $\mathcal{M}^{1\kappa} \kappa^{*\mathfrak{z}1} = \mathcal{M}^{1\kappa} = \mathcal{M}_{\mathcal{N}^{1\kappa}(\varphi)}$.

The theorem now follows from 28.33).

Since $K^{1\mathfrak{z}} M^{1\mathfrak{z}}$ gives $K^{1\mathfrak{z}} I^{1\kappa} 2_{\kappa^*}$, we note that if $\mathcal{M}^{1\kappa^*} \varepsilon_{\kappa}^1 1_{\kappa} \subset \mathcal{M}^{1\kappa^*} \kappa^{*\mathfrak{z}1}$ holds, then these two spaces are necessarily identical, by virtue of 28.1).

III. NON-MODULAR RECIPROGALS OF NON-MODULAR MATRICES

In IV C of his second theory of General Analysis, E. H. Moore studies the problem of the reciprocal of a modular matrix relative to $(\epsilon^1 \ \epsilon^2)$ of the basis $(\mathcal{U} \cdot \mathcal{P}' \cdot \mathcal{P}^2 \cdot \mathcal{E}' \cdot \mathcal{E}^2)$. A necessary and sufficient condition for a modular matrix K^{12} to have a reciprocal according to his definition is that K^{12} be modular relative to $(\epsilon_{\kappa*}^1 \ \epsilon_{\kappa}^2)$.

Consider

$$\mathcal{U} = \mathcal{U} \cdot \mathcal{P}' = \mathcal{P}^2 = \mathcal{P}^{\text{III}} \quad \epsilon^1 = \epsilon^2 = \delta \quad \Theta = (1, \frac{1}{2}, \dots, \frac{1}{2n-1}, \frac{1}{2n}, \dots) \cdot K'^{12} \equiv \delta_{\theta}$$

Then $\epsilon_{\kappa}^1 = \epsilon_{\kappa*}^2 = \delta$ and $\epsilon_{\kappa*}^1 = \epsilon_{\kappa}^2 = \delta_{\theta^2}$. This matrix δ_{θ} is neither modular relative to $(\delta \ \delta)$, nor to $(\delta_{\theta^2} \ \delta_{\theta^2})$, yet there exists a $\lambda \equiv \delta_{\theta^{-1}}$ such that $SK\lambda = \delta$ and $S\lambda K = \delta$.

The present section gives a more general definition for a reciprocal of a T-matrix, retaining, however, the conditions $J^2 K^{12} \lambda^{\epsilon^1} = \epsilon_{\kappa}^1$ and $J^1 \lambda^{\epsilon^1} K^{12} = \epsilon_{\kappa*}^2$. Furthermore, when K^{12} is M^{12} and $M^{1\kappa*} \epsilon_{\kappa}^2$, the new definition gives the results already studied by E. H. Moore. Although the definition is more general, and many results are new, yet the methods used here are essentially the same as E. H. Moore's.

In order to obtain the uniqueness of the reciprocal λ^{ϵ^1} , we assume K^{12} to have the $I^{1\kappa} \epsilon_{\kappa*}^2$ -property, which is weaker than the usual I^{12} -property and is sufficient for our purpose.

The basis for the present study is

$$\mathcal{U}^D \cdot \mathcal{P}' \cdot \mathcal{P}^2 \cdot \mathcal{E}^{1PH} \cdot \mathcal{E}^{2PH} \cdot K'^{12} T'^2 \cdot I'^{1\kappa^2 \kappa*}$$

§ 7

D 31.1r $\lambda^{s1} = r \text{ rec } K^{1s} \equiv \lambda^{s1} \text{ cols } M^{s \times *}. \text{ comp cols } \mathcal{M}^{2 \times *}$

$$J^s K^{1s} \lambda^{s1} = e_{\kappa}^1 \cdot \mathcal{M}_{\cap(1, \lambda)}^{1 \times \kappa} \subset \mathcal{M}^{1 \times \kappa \times 2 \times *}$$

D 31.1x $\lambda^{s1} = x \text{ rec } K^{1s} \equiv \lambda^{s1} \overline{\text{rows}} M^{1 \times}. \text{ comp } \overline{\text{rows}} \mathcal{M}^{1 \times}$

$$J^1 \lambda^{s1} K^{1s} = e_{\kappa^*}^s \cdot \mathcal{M}_{\cap(2, \kappa^*)}^{2 \times \kappa^*} \subset \mathcal{M}^{2 \times \kappa^* \times 1 \times}$$

D 31.2) $\lambda^{s1} = \text{rec } K^{1s} \equiv \lambda^{s1} r \text{ rec } K^{1s} \cdot x \text{ rec } K^{1s}$

If we replace the last condition in D 31.1r by

$$\mathcal{M}^{1 \times} \subset \mathcal{M}^{1 \times \kappa^* \times 2 \times *},$$

then it can be shown that λ^{s1} is M^{s1} , and the second condition, $\lambda^{s1} \text{ comp cols } \mathcal{M}^{2 \times *}$, is redundant. (See Chapter IV.)

When K^{1s} is M^{1s} , the spaces $\mathcal{M}^{1 \times \kappa^* \times 2 \times *}$ and $\mathcal{M}^{1 \times}$ are equal, and the last condition in D 31.1r may be omitted. (See Chapter V.)

If we were interested only in the unique existence of λ_r^{s1} or λ_l^{s1} then the conditions

$$\lambda^{s1} \text{ cols } M^{s \times *}. J^s K^{1s} \lambda^{s1} = e_{\kappa}^1,$$

or

$$\lambda^{s1} \overline{\text{rows}} M^{1 \times}. J^1 \lambda^{s1} K^{1s} = e_{\kappa^*}^s,$$

would be sufficient as a definition. The other conditions are imposed on λ_r^{s1} and λ_l^{s1} so as to secure the mutuality property of the reciprocal, viz. K^{1s} is the reciprocal of λ^{s1} . In order to obtain this property, λ^{s1} must have the property $I^{s \times 1 \times *}$, which condition is proved when λ^{s1} exists.

If K^{1s} is assumed to be I^{1s} instead of $I^{1 \times \kappa^* \times *}$, then λ^{s1} will retain its mutuality property, in case the last conditions in D 31.1r and D 31.1x are respectively replaced by

$$\mathcal{M}_{\cap(1, \lambda)}^{1 \times \kappa} \subset \mathcal{M}^{1 \times \kappa^* \times 2 \times *} \quad \mathcal{M}_{\cap(2, \kappa)}^{2 \times \kappa} \subset \mathcal{M}^{2 \times \kappa^* \times 1 \times}.$$

The following theorem states properties of λ_r^{s1} . Necessary and sufficient conditions of its existence are secured.

$$T \quad 31.11) \quad \mathfrak{A} \lambda^{s1} = r \text{ rec } K^{1s} \quad .).$$

$$\begin{aligned} 31.12) \quad \mathfrak{A} \lambda^{s1} = r \text{ rec } K^{1s} \quad \sim. \quad e_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \mathcal{M}'_{\kappa*} e_{\kappa}^{1\kappa} \subset \mathcal{M}'_{\kappa*} \kappa^{*2\kappa} \\ \sim. \quad e_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot I^{1\kappa 1\kappa*} \\ .).$$

$$31.13) \quad \lambda^{s1} = r \text{ rec } K^{1s} \quad .).$$

$$31.131) \quad \lambda^{s1} \text{ cols } \mathcal{M}^{2\kappa* \kappa' \kappa}$$

$$31.132) \quad J^s \lambda^{*1s} \mathcal{M}_{\cap(2\kappa* 2\kappa)}^2 = \mathcal{M}'_{\kappa \kappa^{*2\kappa*}} = \mathcal{M}'_{\cap(1\kappa' 1\kappa)}$$

$$31.2) \quad \mathfrak{A} \lambda_{\ell}^{s1} \cdot \mathfrak{A} \lambda_r^{s1} \quad .).$$

To prove 31.11), let us suppose $\mathfrak{A}(\lambda_1^{s1} \cdot \lambda_2^{s1})$. Then $J^s K^{1s} \lambda_1^{s1} = e_{\kappa}^1 = J^s K^{1s} \lambda_2^{s1}$. Thus $J^s K^{1s}(\lambda_1^{s1} - \lambda_2^{s1}) = O^{11}$. Now $K^{1s} \text{ comp } \overline{\text{rows}} \mathcal{M}^{2\kappa*}$, and $(\lambda_1^{s1} - \lambda_2^{s1}) \text{ cols } M^{2\kappa*}$. Hence $\lambda_1^{s1} = \lambda_2^{s1}$. 31.12) follows from 26.1), 26.2), and 28.33). To prove 31.131), we have $\lambda^{s1} \text{ cols } M^{2\kappa*}$. Furthermore, from

$$J^s K^{1s} \lambda^{s1}(.p^1) = e_{\kappa}^1(.p^1) \quad (p^1)$$

we see that $\lambda^{s1} \text{ cols } \mathcal{M}^{2\kappa* \kappa' \kappa}$. The first equality in 31.132) follows from 28.2) whereas the second equality follows from 28.33) and 28.2) by virtue of 31.12).

To prove 31.2), we make use of 31.131) and its hermitian parity. Since $K^{1s} I^{1\kappa 2\kappa*}$, we obtain

$$\begin{aligned} \lambda_{\ell}^{s1} &= J^1 \lambda_{\ell}^{s1} e_{\kappa}^1 = J^1 \lambda_{\ell}^{s1} (J^s K^{1s} \lambda_r^{s1}) \\ &= J^s (J_{\ell}^1 \lambda^{s1} K^{1s}) \lambda_r^{s1} = J^s e_{\kappa}^{2\kappa*} \lambda^{s1} = \lambda_r^{s1} \quad . \end{aligned}$$

Thus 31.2) is established.

From 31.2) it follows that whenever a reciprocal λ^{s1} exists, it is the unique reciprocal.

Properties of reciprocal λ^{s1} are stated in the following theorem:

$$T \ 32.1) \ \lambda^{s1} = \text{rec } K^{1s} \ .) . \ \lambda^{s1} \text{ type } \mathcal{M}^{2_{K^*} \times 1_K} . \overline{\mathcal{M}^{1_K \times 2_{K^*}}}$$

$$32.2) \ \lambda^{s1} = \text{rec } K^{1s} \ .):$$

$$32.21) \ J^1(J^s K^{1s} \lambda^{s1}) K^{1s} = J^s K^{1s} (J^1 \lambda^{s1} K^{1s}) = K^{1s}$$

$$J^s (J^1 \lambda^{s1} K^{1s}) \lambda^{s1} = J^1 \lambda^{s1} (J^s K^{1s} \lambda^{s1}) = \lambda^{s1}$$

$$32.22) \ \text{In general, } K^{1s} \sim M^{1_{K^*} \times 2_K} . \ \lambda^{s1} \sim M^{2_{K^*} \times 1_K}$$

$$32.23) \ \varepsilon_K^1 = \varepsilon_{\lambda^*}^1 \ , \ \varepsilon_{K^*}^s = \varepsilon_{\lambda}^s$$

$$32.24) \ \mathcal{M}'_{\cap(1_K \times 1_{K^*})} = \mathcal{M}'^{1_{K^*} \times 2_{\lambda}} \subset \mathcal{M}'^{1_{K^*}} = \mathcal{M}'^{1_K}$$

$$\mathcal{M}^2_{\cap(2_{K^*} \times 2_K)} = \mathcal{M}^{2_{\lambda} \times 1_{K^*}} \subset \mathcal{M}^{2_{\lambda}} = \mathcal{M}^{2_{K^*}}$$

$$32.25) \ \mathcal{M}'_{\cap(1_{K^*} \times 1_{\lambda})} = \mathcal{M}'^{1_K \times 2_{K^*}} \cdot \mathcal{M}^2_{\cap(2_{\lambda} \times 2_{K^*})} = \mathcal{M}^{2_{K^*} \times 1_K}$$

$$32.26) \ J^{s_K} K^{1s} = J^s \lambda^{*1s} \text{ on } \mathcal{M}^2_{\cap(2_{K^*} \times 2_K)}$$

$$J^{1_{K^*}} K^{*s1} = J^1 \lambda^{s1} \text{ on } \mathcal{M}'_{\cap(1_K \times 1_{K^*})}$$

$$32.27) \ \lambda^{s1} \sim I^{2_{\lambda} \times 1_{K^*}}$$

$$32.28) \ K^{1s} \text{ comp cols } \mathcal{M}' \ .) . \ \lambda^{s1} \text{ comp rows } \mathcal{M}'$$

$$K^{1s} \text{ comp rows } \mathcal{M}^2 \ .) . \ \lambda^{s1} \text{ comp cols } \mathcal{M}^2$$

$$32.29) \ \lambda^{*1s} = \text{rec } K^{*s1} \ . \ K^{1s} = \text{rec } \lambda^{s1}$$

$$32.3) \ \exists \ \lambda^{s1} = \text{rec } K^{1s} \ . \sim . \ \varepsilon_K' \text{ cols } M^{1_{K^*} \times 1_K} \cdot \varepsilon_{K^*}^2 \text{ rows } M^{2_K \times 1_{K^*}} \sim .$$

$$\sim \cdot \varepsilon_{\kappa}^{\prime} \text{ cols } M^{\prime \kappa*} \cdot \varepsilon_{\kappa*}^2 \text{ rows } M^{2\kappa*} \cdot \mathcal{M}^{\prime \kappa* \varepsilon_{\kappa}^{\prime} \kappa} \subset \mathcal{M}^{\prime \kappa* \kappa* 2\kappa} \cdot \mathcal{M}^{2\kappa \varepsilon_{\kappa*}^2 2\kappa*} \subset \mathcal{M}^{2\kappa \kappa' \kappa*}$$

$$\cdot) \cdot J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1 = J^{2\kappa} \varepsilon_{\kappa*}^2 K^{*21} = \text{rec } K^{12}$$

$$32.4) \lambda^{21} = \text{rec } K^{12} \cdot) \cdot K^{12} = J^{2\lambda*} \lambda^{*12} \varepsilon_{\lambda}^2 = J^{1\lambda} \varepsilon_{\lambda*}^1 \lambda^{*12}$$

In 32.3), we have necessary and sufficient conditions for the existence of a reciprocal, λ^{21} , of K^{12} . The formula $\lambda^{21} = J^{1\kappa*} K^{*21} \varepsilon_{\kappa}^1 = J^{2\kappa} \varepsilon_{\kappa*}^2 K^{*21}$ is effective as a reciprocal only when necessary and sufficient conditions are satisfied.

As to the proof of the theorem, we note that 32.1) is obvious from 31.131) and its hermitian parity for left reciprocals.

32.21) follows from the definitions of reciprocals, and IV A, T II. That, in general, $K^{12} \sim M^{1\kappa* 2\kappa}$ and $\lambda^{21} \sim M^{2\kappa* 1\lambda}$ can be shown by the example given on p. 47.¹ To prove 32.23), we use 26.4) and get $\mathcal{M}^{2\lambda} = \mathcal{M}^{2\kappa*}$. Hence by IV B theory, $\varepsilon_{\lambda}^2 = \varepsilon_{\kappa*}^2$. By hermitian parity, we secure $\varepsilon_{\lambda*}^1 = \varepsilon_{\kappa}^1$.

In 32.24), we note that $\mathcal{M}^{\prime \kappa} = \mathcal{M}^{\prime \lambda*} \supset \mathcal{M}^{\prime \lambda* \lambda} \supset \mathcal{M}^{\prime \lambda*}$ is obvious. To show the equality $\mathcal{M}_{\cap(\prime \kappa' \kappa*)}^{\prime} = \mathcal{M}^{\prime \lambda* \lambda} \supset \mathcal{M}^{\prime \lambda*}$, we use 32.23) and the hermitian parity of 31.132), and secure

$$J^1 \lambda^{21} \mathcal{M}_{\cap(\prime \kappa' \kappa*)}^1 = \mathcal{M}_{\cap(2\kappa* 2\kappa*)}^2 = \mathcal{M}_{\cap(2\lambda 2\lambda*)}^2.$$

Now λ^{21} is of $T^{2\kappa* 1\kappa}$ and hence is of T^{21} . Applying the results in 22.51) to λ^{21} , we obtain

$$J^1 \lambda^{21} \mathcal{M}^{\prime \lambda* \lambda} \supset \mathcal{M}^{\prime \lambda*} = J^{1\lambda*} \lambda^{21} \mathcal{M}^{\prime \lambda* \lambda} \supset \mathcal{M}^{\prime \lambda*} = \mathcal{M}_{\cap(2\lambda 2\lambda*)}^2.$$

Consequently, $J^1 \lambda^{21} \mathcal{M}_{\cap(\prime \kappa' \kappa*)}^1 = J^1 \lambda^{21} \mathcal{M}^{\prime \lambda* \lambda} \supset \mathcal{M}^{\prime \lambda*}$. Now $\varepsilon_{\kappa}^1 = \varepsilon_{\lambda*}^1$ gives $\mathcal{M}_{\cap(\prime \kappa' \kappa*)}^1 \subset \mathcal{M}^{\prime \lambda*}$, and since $\lambda^{21} \text{ comp } \overline{\text{rows}} \mathcal{M}^{\prime \lambda*}$, we obtain $\mathcal{M}_{\cap(\prime \kappa' \kappa*)}^1 = \mathcal{M}^{\prime \lambda* \lambda} \supset \mathcal{M}^{\prime \lambda*}$. By parity, we obtain the second part of 32.24).

¹ To prove the existence of a reciprocal for this example, we should make use of T 34.4) on p. 58, and T 25.52) on p. 30.

To prove 32.25), we first observe from the definition of λ^{21} that $\mathcal{M}_{\cap(1_k' 1_\lambda)}^{1'} \subset \mathcal{M}_{\cap(1_k' 1_{k^*} 2_{k^*})}^{1'}$. Secondly, from 26.1), 28.2), and 32.23), we have $\mathcal{M}_{\cap(1_k' 1_{k^*} 2_{k^*})}^{1'} = \mathcal{M}_{\cap(1_{\lambda^*} 1_\lambda)}^{1'}$.

32.26) follows from 28.32) since $\varepsilon_{\kappa}^1 = \varepsilon_{\lambda^*}^1$.

To prove 32.27), let us take any pair $\mu_{\cap(1_k' 1_{k^*})}^1 \cdot \mu_{\cap(2_{k^*} 2_\kappa)}^2$. Then by IV A, T II, 6) and T I, 3), III A, T II, 10), and 32.26) just established, we have

$$\begin{aligned} J^1(J^2 \overline{\mu_{\cap(2_{k^*} 2_\kappa)}^2} \lambda^{21}) \mu_{\cap(1_k' 1_{k^*})}^1 &= J^1(J^2 \overline{\mu_{\cap(2_{k^*} 2_\kappa)}^2} K^{*21}) \mu_{\cap(1_k' 1_{k^*})}^1 \\ &= J^1 \overline{J^2 \mu_{\cap(2_{k^*} 2_\kappa)}^2} K^{*21} \mu_{\cap(1_k' 1_{k^*})}^1 = (J^1 \overline{J^2 \mu_{\cap(2_{k^*} 2_\kappa)}^2}) K^{*21} \mu_{\cap(1_k' 1_{k^*})}^1 \end{aligned}$$

also

$$\begin{aligned} J^2 \overline{\mu_{\cap(2_{k^*} 2_\kappa)}^2} (J^1 \lambda^{21} \mu_{\cap(1_k' 1_{k^*})}^1) &= J^2 \overline{\mu_{\cap(2_{k^*} 2_\kappa)}^2} (J^1 \overline{K^{*21}} \mu_{\cap(1_k' 1_{k^*})}^1) \\ &= J^2 \overline{K^{*21} \mu_{\cap(2_{k^*} 2_\kappa)}^2} J^1 \overline{\mu_{\cap(1_k' 1_{k^*})}^1} = (J^1 \overline{J^2 \mu_{\cap(2_{k^*} 2_\kappa)}^2}) K^{*21} \mu_{\cap(1_k' 1_{k^*})}^1. \end{aligned}$$

Now, by hypothesis, $K^{12} \in I^{1_k 2_{k^*}}$. Therefore, by 25.3), we conclude that $J^1(J^2 \overline{\mu_{\cap(2_{k^*} 2_\kappa)}^2} \lambda^{21}) \mu_{\cap(1_k' 1_{k^*})}^1 = J^2 \overline{\mu_{\cap(2_{k^*} 2_\kappa)}^2} J^1 \lambda^{21} \mu_{\cap(1_k' 1_{k^*})}^1$. Thus by 32.24), we secure $\lambda^{21} \in I^{2_{\lambda^*} 1_{\lambda^*}}$.

To prove 32.28), we use 32.23). Thus, if K^{12} is complete by cols $\mathcal{M}^1$, we have, by IV A, T III, $\varepsilon^1 = \varepsilon_{\kappa}^1 = \varepsilon_{\lambda^*}^1$. Hence $\lambda^{21} \text{ comp rows } \mathcal{M}^1$. The second part is secured by parity.

The first part in 32.29) is obvious. The second part follows from 32.27), 32.23), and 32.24).

32.3) follows immediately from 31.12) and its hermitian parity. 32.4) follows from 32.29) and 32.3).

§ 8

For the study of classical reciprocals, we add the con-

ditions

$$\mathcal{M}_{\cap(1, \kappa^*)}^1 \subset \mathcal{M}_{\cap(1, \kappa^*)}^1 \cdot \mathcal{M}_{\cap(2, 2\kappa)}^2 \subset \mathcal{M}_{\cap(2\kappa^*, 2\kappa)}^2$$

to the basis on p. 47. These two conditions together with the $I^{1\kappa^* 2\kappa^*}$ -property of K^{12} gives, by 25.42), $K^{12} I^{12}$.

$$D \ 32.1r \quad \lambda^{21} = r \operatorname{rec} K^{12} \ . \equiv . \lambda^{21} \operatorname{cols} M^2 \ . \ J^2 K^{12} \lambda^{21} = \epsilon^1$$

$$\cdot J^{1\lambda} \lambda^{21} \mathcal{M}_{\cap(1, \kappa^*)}^1 = \mathcal{M}_{\cap(2\kappa^*, 2\kappa)}^2$$

$$D \ 32.1\lambda \quad \lambda^{21} = \lambda \operatorname{rec} K^{12} \ . \equiv . \lambda^{21} \operatorname{rows} M^1 \ . \ J^1 \lambda^{21} K^{12} = \epsilon^2$$

$$\cdot J^{2\lambda^*} \lambda^{*12} \mathcal{M}_{\cap(2\kappa^*, 2\kappa)}^2 = \mathcal{M}_{\cap(1, \kappa^*)}^1$$

$$D \ 32.2) \quad \lambda^{21} = \operatorname{rec} K^{12} \ . \equiv . \lambda^{21} r \operatorname{rec} K^{12} \cdot \lambda \operatorname{rec} K^{12}$$

For convenience, $\lambda_{\circ}^{21}$, $\lambda_{\circ}^{21}$, and $\lambda_{\circ}^{21}$ shall indicate $r \operatorname{rec} K^{12}$, $\lambda \operatorname{rec} K^{12}$, and $\operatorname{rec} K^{12}$ respectively.

If we are interested only in the uniqueness of $\operatorname{rec} K^{12}$, then we may use the following definitions:

$$\lambda^{21} = r \operatorname{rec} K^{12} \ . \equiv . \lambda^{21} \operatorname{cols} M^2 \ . \ J^2 K^{12} \lambda^{21} = \epsilon^1,$$

$$\lambda^{21} = \lambda \operatorname{rec} K^{12} \ . \equiv . \lambda^{21} \operatorname{rows} M^1 \ . \ J^1 \lambda^{21} K^{12} = \epsilon^2$$

These definitions, however, do not give us the mutuality property. Thus the last conditions in D 32.1r and D 32.1λ are added in such wise that when $\lambda_{\circ}^{21}$ exists, it has the property I^{21} , and the mutuality property can be proved.

$$T \ 33.1) \quad \lambda^{21} = r \operatorname{rec} K^{12} \ .) \cdot \lambda^{21} \operatorname{cols} \mathcal{M}^{2\kappa^*} \cdot \mathcal{M}^{2\lambda} \supset \mathcal{M}_{\cap(2\kappa^*, 2\kappa)}^2$$

$$33.2) \quad \mathbb{E} \lambda_{\circ}^{21} \cdot \mathbb{E} \lambda_{\circ}^{21} \cdot) \cdot \mathbb{E} \lambda_{\circ}^{21} = \mathbb{E} \lambda_{\circ}^{21} = \mathbb{E} \lambda_{\circ}^{21}$$

$$33.3) \quad \mathbb{E} \lambda^{21} = r \operatorname{rec} K^{12} \ .) \cdot$$

$$33.31) \quad K^{12} \text{ comp cols } \mathcal{M}'$$

$$33.32) \quad \epsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \text{comp cols } \mathcal{M}'^{\kappa*}$$

$$33.33) \quad \exists! \lambda_1^{21} \text{ (viz. } J^{1\kappa*} K^{*21} \epsilon_{\kappa}^1) \text{ cols } M^{2\kappa*} \exists \cdot J^2 K^{12} \lambda_1^{21} = e^1 \\ \cdot \lambda_1^{21} \text{ comp cols } \mathcal{M}'^{2\kappa*} \cdot \mathcal{M}'_{\alpha'(\lambda_1)} \subset \mathcal{M}'^{\kappa* 2\kappa}$$

$$33.4) \quad (1) \exists \lambda^{21} = r \text{ rec } K^{12} \sim (2) K^{12} \text{ comp cols } \mathcal{M}' \cdot \epsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot J^{1\kappa*}$$

$$\sim (3) K^{12} \text{ comp cols } \mathcal{M}' \cdot \epsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \mathcal{M}'^{\kappa*} \epsilon_{\kappa}^1 \subset \mathcal{M}'^{\kappa* 2\kappa}$$

$$\cdot) \cdot (4) J^{1\kappa*} K^{*21} \epsilon_{\kappa}^1 = r \text{ rec } K^{12}$$

$$33.5) \quad \exists \lambda^{21} = \text{rec } K^{12} \sim$$

$$\sim K^{12} \text{ comp cols } \mathcal{M}' \cdot \text{comp rows } \mathcal{M}'^2 \cdot \epsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot I^{1\kappa*} \epsilon_{\kappa}^2 \text{ rows } M^{2\kappa} \cdot I^{2\kappa* 2\kappa}$$

$$\sim K^{12} \text{ comp cols } \mathcal{M}' \cdot \text{comp rows } \mathcal{M}'^2 \cdot \epsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \epsilon_{\kappa}^2 \text{ rows } M^{2\kappa} \\ \cdot \mathcal{M}'^{\kappa*} \epsilon_{\kappa}^1 \subset \mathcal{M}'^{\kappa* 2\kappa} \cdot \mathcal{M}'^{2\kappa} \epsilon_{\kappa}^2 \subset \mathcal{M}'^{2\kappa \kappa'}$$

$$\cdot) \cdot J^{1\kappa*} K^{*21} \epsilon_{\kappa}^1 = J^{2\kappa} \epsilon_{\kappa}^2 K^{*21} = \text{rec } K^{12}$$

$$33.6) \quad \lambda^{21} = \text{rec } K^{12} \cdot) \cdot 33.61) \quad \lambda^{21} = \text{rec } K^{12}$$

$$33.62) \quad \lambda^{*12} = \text{rec } K^{*21}$$

$$33.63) \quad \lambda^{21} I^{21}$$

$$33.64) \quad K^{12} = \text{rec } \lambda^{21}$$

In 33.1) the first property is evident from the definition. From IV A theory, we note that $\lambda^{21} M^{2\lambda 1\lambda}$ and $\mathcal{M}'^{2\lambda} = J^{1\lambda} \lambda^{21} \mathcal{M}'^{1\lambda}$. Hence from the last condition in D 32.1r, we have the second property.

To prove 33.2), we note that $J^2 (J^1 \lambda_l^{21} K^{12}) \lambda_r^{21} = J^2 \epsilon^2 \lambda_r^{21} = \lambda_r^{21}$, and $J^1 \lambda_l^{21} J^2 K^{12} \lambda_r^{21} = J^1 \lambda_l^{21} \epsilon^1 = \lambda_l^{21}$. From 33.1) and its hermitian parity, we find that λ_l^{21} is by $\overline{\text{rows}} \mathcal{M}'^{\kappa* 2}$ and λ_r^{21} is

by cols $\mathcal{M}^{2 \times 1}$. Since $K^{12} \perp^{12}$, we have $J^2(J^1 \lambda_i^{21} K^{12}) \lambda_r^{21} = J^1 \lambda_i^{21} J^2 K^{12} \lambda_r^{21}$, from which the uniqueness follows.

To prove 33.31), consider $J^1 \bar{\mu}^T K^{12} = 0^2$. Then μ^1 is a vector in $\mathcal{M}^{1 \times 2}$. Since λ^{21} cols $\mathcal{M}^{2 \times 1}$ and $K^{12} \perp^{12}$, we have

$$0^1 = J^2(J^1 \bar{\mu}^T K^{12}) \lambda^{21} = J^1 \bar{\mu}^T J^2 K^{12} \lambda^{21} = J^1 \bar{\mu}^T \varepsilon^1 = \bar{\mu}^T,$$

whence K^{12} comp cols $\mathcal{M}^1$.

From 33.31) and D 32.1r, we have $J^2 K^{12} \lambda^{21} = \varepsilon_\kappa^1$. Hence by 26.1), ε_κ^1 cols $M^{1 \times *}$. To prove ε_κ^1 comp cols $\mathcal{M}^{1 \times *}$, consider $J^{1 \times *} \bar{\mu}^{1 \times *} \varepsilon_\kappa^1 = 0^1$. Then $0^1 = J^{1 \times *} \bar{\mu}^{1 \times *} J^2 K^{12} \lambda^{21} = J^2(J^{1 \times *} \bar{\mu}^{1 \times *} K^{12}) \lambda^{21}$ since $K^{12} M^{1 \times *} \perp^{12}$. Furthermore, on account of the fact that λ^{21} is $M^{21 \times}$, it follows $0^2 = J^{1 \times *} J^2(J^{1 \times *} \bar{\mu}^{1 \times *} K^{12}) \lambda^{21} \lambda^{*12} = J^2 J^{1 \times *} \bar{\mu}^{1 \times *} K^{12} \varepsilon_\lambda^2$, which by 21.2) and 33.1) gives $J^{1 \times *} \bar{\mu}^{1 \times *} K^{12} = 0^2$. Since K^{12} is complete by cols $\mathcal{M}^{1 \times *}$, we have $\mu^{1 \times *} = 0^1$, which proves 33.32).

To prove 33.33), define $\lambda_1^{21} \equiv J^2 \varepsilon_{\kappa*}^2 \lambda^{21}$. Then λ_1^{21} is evidently by cols $M^{2 \times *}$. Furthermore, since $\varepsilon_{\kappa*}^2 M^{22}$, we have

$$J^2 K^{12} \lambda_1^{21} = J^2 K^{12} (J^2 \varepsilon_{\kappa*}^2 \lambda^{21}) = J^2 (J^2 K^{12} \varepsilon_{\kappa*}^2) \lambda^{21} = J^2 K^{12} \lambda^{21} = \varepsilon^1.$$

Thus by 33.31), 33.32), and 26.2), λ_1^{21} is unique and equals $J^{1 \times *} K^{*21} \varepsilon_\kappa^1$. We now prove the following two lemmas:

$$L 1) \quad \mathcal{M}^{1 \times 1} \subset \mathcal{M}^{1 \times}$$

$$L 2) \quad J^2 \lambda_1^{*12} \mathcal{M}_{\cap(2_{\kappa*} 2_{\kappa})}^2 = \mathcal{M}_{\cap(1_{\kappa} 1_{\lambda})}^1$$

By definition of λ_1^{21} , we have

$$\begin{aligned} \varepsilon_{\lambda_1}^1 &= J^2 \lambda_1^{*12} \lambda_1^{21} = J^2 (J^2 \lambda^{*12} \varepsilon_{\kappa*}^2) \lambda_1^{21} = J^2 \lambda^{*12} (J^2 \varepsilon_{\kappa*}^2 \lambda_1^{21}) \\ &= J^2 \lambda^{*12} \lambda_1^{21} = J^2 J^2 \lambda^{*12} \varepsilon_{\kappa*}^2 \lambda^{21}. \end{aligned}$$

Since $\lambda^{*12} M^{1 \times 2}$ and $\varepsilon_{\kappa*}^2 M^{22}$, we have by the composition theorem

of modularity, $e_{\lambda_1}^1 M^{1\lambda_1}$, from which L 1) follows.

From the last condition in D 32.1r, we secure

$$J^2 \lambda^{*12} J^{1\lambda} \lambda^{21} \mathcal{M}'_{\alpha(\lambda_1 \lambda_2)} = J^2 \lambda^{*12} \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)}.$$

On the left hand side, we have

$$J^2 \lambda^{*12} J^{1\lambda} \lambda^{21} \mathcal{M}'_{\alpha(\lambda_1 \lambda_2)} = J^{1\lambda} (J^2 \lambda^{*12} \lambda^{21}) \mathcal{M}'_{\alpha(\lambda_1 \lambda_2)} = \mathcal{M}'_{\alpha(\lambda_1 \lambda_2)},$$

whereas on the right hand side, we have

$$J^2 \lambda^{*12} (J^2 e_{\kappa^*}^2 \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)}) = J^2 (J^2 \lambda^{*12} e_{\kappa^*}^2) \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)} = J^2 \lambda_1^{*12} \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)}.$$

Thus L 2) is demonstrated.

From L 1) and L 2), we have $J^2 \lambda_1^{*12} \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)} > \mathcal{M}'_{\alpha(\lambda_1 \lambda_2)}$.

Thus by 33.32), 28.2), and 28.33), theorem 33.33) is proved.

For 33.4), we shall show (1) $\rightarrow$ (2) $\rightarrow$ (1). That (1) $\rightarrow$ (2) follows from 33.3) and 28.33). To prove (2) $\rightarrow$ (1), let us define $\lambda^{21} \equiv J^{1\kappa^*} K^{*21} e_{\kappa}^1$. Then by 26.2) and K^{12} comp cols $\mathcal{M}'$, we secure λ^{21} cols M^2 . $J^2 K^{12} \lambda^{21} = e^1$. The last condition in D 31.1r) follows from 28.33) and 28.31).

That (2) is equivalent to (3) follows from 28.33).

33.5) is the combined result of 33.4) and its hermitian parity for λ_l^{21} .

From 33.5) we find that $e_{\kappa}^1 = e^1$ and $e_{\kappa^*}^2 = e^2$. Thus λ^{21} is of $T^{2\kappa^* 1\kappa}$, and $J^2 K^{12} \lambda^{21} = e_{\kappa}^1$. $J^1 \lambda^{21} K^{12} = e_{\kappa^*}^2$. Furthermore, by 28.33) λ^{21} comp cols $\mathcal{M}^{2\kappa^*}$ comp rows $\mathcal{M}^{1\kappa}$. $\mathcal{M}'_{\alpha(\lambda_1 \lambda_2)} < \mathcal{M}^{1\kappa} \kappa^* 2\kappa^*$. $\mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)} < \mathcal{M}^{2\kappa^* \kappa} 1\kappa$. This proves 33.61).

33.62) is evident from the definitions.

To prove 33.63), we note from 33.61) and 32.27) that $\lambda^{21} I^{2\lambda_1 \lambda^*}$. Further, by 32.23), $e_{\lambda^*}^1 = e_{\kappa}^1 = e^1$, and $e_{\lambda}^2 = e_{\kappa^*}^2 = e^2$. Thus $\lambda^{21} I^{21}$.

To prove $K^{12} = \text{rec } \lambda^{21}$, we shall prove $K^{12} = r \text{ rec } \lambda^{21}$, from which we may deduce $K^{12} = \lambda \text{ rec } \lambda^{21}$ by hermitian parity. Now it is evident that K^{12} cols M^1 . $J^1 \lambda^{21} K^{12} = \epsilon^2$. It remains to prove

$$J^{2K} K^{12} \mathcal{M}_{\cap(2\lambda, 2K)}^2 = \mathcal{M}_{\cap(1\lambda^*, 1\lambda)}^1.$$

From 33.61) and 32.23), we have $\epsilon_\lambda^2 = \epsilon_{\lambda^*}^2$. Thus $J^{2K} K^{12} \mathcal{M}_{\cap(2\lambda, 2K)}^2 = J^{2K} K^{12} \mathcal{M}_{\cap(2K^*, 2K)}^2$. By 22.54), 33.61), and 32.25), we have the result.

§ 9

Theorem 34) is devoted to the study of necessary and sufficient conditions for a matrix λ^{21} to be a reciprocal of K^{12} .

T 34) $\lambda^{21} = \text{rec } K^{12}$

:~: 34.1) $\lambda^{21} T^{2K^* 1K}$. comp cols $\mathcal{M}^{2K^*}$. comp rows $\mathcal{M}^{1K}$

$$\cdot J^2 K^{12} \lambda^{21} = \epsilon_K^1$$

$$\cdot \mathcal{M}_{\cap(1\lambda^*, 1\lambda)}^1 \subset \mathcal{M}^{1K^* K^* 2K^*} \cdot \mathcal{M}_{\cap(2\lambda, 2\lambda^*)}^2 \subset \mathcal{M}^{2K^* K^* 1K}$$

:~: 34.2) $\lambda^{21} T^{2K^* 1K}$. comp cols $\mathcal{M}^{2K^*}$. comp rows $\mathcal{M}^{1K}$

$$\cdot J^2 K^{12} \lambda^{21} = \epsilon_K^1$$

$$\cdot J^2 \lambda^{*1/2} \mathcal{M}^{2\lambda^* \lambda^* 1K^*} \subset \mathcal{M}^{1K^* K^* 2K^*} \cdot J^1 \lambda^{2/} \mathcal{M}^{1\lambda^* \lambda^* 2\lambda} \subset \mathcal{M}^{2K^* K^* 1K}$$

:~: 34.3) $\lambda^{21} T^{21}$. $\mathcal{M}^{2\lambda} = \mathcal{M}^{2K^*}$. $\mathcal{M}^{1\lambda^*} = \mathcal{M}^{1K}$. $J^2 K^{12} \lambda^{21} = \epsilon_K^1$

$$\cdot J^{1\lambda} \lambda^{2/} \mathcal{M}_{\cap(1\lambda^*, 1\lambda)}^1 \subset \mathcal{M}_{\cap(2K^*, 2K)}^2 \cdot J^{2\lambda^*} \lambda^{*1/2} \mathcal{M}_{\cap(2\lambda, 2\lambda^*)}^2 \subset \mathcal{M}_{\cap(1K^*, 1K)}^1$$

$$:\sim: 34.4) \quad \lambda^{E1} T^{E1} \cdot I^{E\lambda} \cdot \mathcal{M}^{2\lambda} = \mathcal{M}^{2K*} \cdot \mathcal{M}^{1\lambda*} = \mathcal{M}^{1K}$$

$$\cdot J^{EK^{1E}} \lambda^{E1} = e_{\kappa}^1 \cdot J^1 \lambda^{E1} K^{1E} = e_{\kappa*}^E.$$

$$:\sim: 34.5) \quad \lambda^{E1} T^{E1} \cdot I^{2\lambda} \cdot \mathcal{M}^{2\lambda} = \mathcal{M}^{2K*} \cdot \mathcal{M}^{1\lambda*} = \mathcal{M}^{1K} \cdot \overline{\text{rows}} \mathcal{M}^{1K} K^{*2K*}$$

$$\cdot J^{EK^{1E}} \lambda^{E1} = e_{\kappa}^1$$

$$:\sim: 34.6) \quad \lambda^{E1} T^{EK*} \cdot I^{E\lambda} \cdot \text{comp cols } \mathcal{M}^{2K*} \cdot \text{comp } \overline{\text{rows}} \mathcal{M}^{1K}$$

$$\cdot \text{type } \mathcal{M}^{2K*} K^{1K} \cdot \overline{\mathcal{M}^{1K} K^{*2K*}} \cdot J^1 \lambda^{E1} J^{EK^{1E}} \lambda^{E1} = \lambda^{E1}$$

$$:\sim: 34.7) \quad \lambda^{E1} T^{EK*} \cdot \text{cols } \mathcal{M}^{2K*} K^{1K} \cdot \text{comp cols } \mathcal{M}^{2K*} \cdot \text{comp } \overline{\text{rows}} \mathcal{M}^{1K}$$

$$: \mathcal{M}_{\cap(1\lambda*1\lambda)}^{1K} \subset \mathcal{M}^{1K} K^{*2K*} : \mathcal{M}_{\cap(2\lambda 2\lambda*)}^2 \subset \mathcal{M}^{2K*} K^{1K}$$

$$: J^{EK^{1E}} \mu_{\cap(2K*2\lambda*)}^E = \mu_{\cap(1K1K*)}^1 \cdot J^1 \lambda^{E1} \mu_{\cap(1K1K*)}^1 = \mu_{\cap(2K*2\lambda*)}^E$$

$$:\sim: 34.8) \quad \lambda^{E1} T^{E1} \cdot I^{E\lambda} \cdot \text{type } \mathcal{M}^{2K*} K^{1K} \cdot \overline{\mathcal{M}^{1K} K^{*2K*}}$$

$$: \mathcal{M}^{2\lambda} = \mathcal{M}^{2K*} : \mathcal{M}^{1\lambda*} = \mathcal{M}^{1K}$$

$$: J^{EK^{1E}} \mu_{\cap(2\lambda 2\lambda*)}^E = \mu_{\cap(1K1K*)}^1 \cdot J^1 \lambda^{E1} \mu_{\cap(1K1K*)}^1 = \mu_{\cap(2K*2\lambda*)}^2$$

The scheme of demonstration of this theorem will be as follows: $0) \rightarrow 34.4) \rightarrow 34.1) \rightarrow 34.2) \rightarrow 34.3) \rightarrow 0$; $0) \rightarrow 34.5) \rightarrow 34.6) \rightarrow 34.4$; $0) \rightarrow 34.7) \rightarrow 34.1$; $0) \rightarrow 34.8) \rightarrow 34.6$.

That $0) \rightarrow 34.4)$ follows from 32.23) and 32.27).

For $34.4) \rightarrow 34.1)$, we use 26.4), and obtain

$$\lambda^{E1} T^{EK*} \cdot \text{comp cols } \mathcal{M}^{2K*} \cdot \text{comp } \overline{\text{rows}} \mathcal{M}^{1K}$$

It remains to prove $\mathcal{M}_{\cap(1\lambda*1\lambda)}^{1K} \subset \mathcal{M}^{1K} K^{*2K*} \cdot \mathcal{M}_{\cap(2K*2K)}^2 \subset \mathcal{M}^{2K*} K^{1K}$. Consider any $\mu_{\cap(1\lambda*1\lambda)}^1$, which is certainly M^{1K} , for by hypothesis, $\mathcal{M}^{1\lambda*} = \mathcal{M}^{1K}$. By 22.51),

$$\exists \mu^{E\lambda} \lambda^{*1\lambda*} \ni \mu_{\cap(1\lambda*1\lambda)}^1 = J^E \lambda^{*1E} \mu_{\lambda}^{E\lambda} \lambda^{*1\lambda*},$$

from which, we have $J^1 K^{*21} u_{\alpha(\lambda^* \lambda)}^1 = J^1 K^{*21} (J^2 \lambda^{*12} u_{\alpha \lambda^* \lambda^*}^{2\lambda^* 1\lambda^*}) =$
 $= J^2 (J^1 K^{*21} \lambda^{*12}) u_{\alpha \lambda^* \lambda^*}^{2\lambda^* 1\lambda^*} = J^2 \epsilon_{\kappa^*}^2 u_{\alpha \lambda^* \lambda^*}^{2\lambda^* 1\lambda^*} = u_{\alpha \lambda^* \lambda^*}^{2\lambda^* 1\lambda^*} M^{2\kappa^*}$. Hence
 $\mathcal{M}'_{\alpha(\lambda^* \lambda)} < \mathcal{M}^{1\kappa^* \kappa^* 2\kappa^*}$. Similarly, we secure the last condition
 in 34.1).

34.1) $\rightarrow$ 34.2): Applying 22.51) to λ^{21} , we have

$$\mathcal{M}'_{\alpha(\lambda^* \lambda)} = J^2 \lambda^{*12} \mathcal{M}^{2\lambda^* \lambda^* 1\lambda^*}. \quad \mathcal{M}^2_{\alpha(2\lambda^* 2\lambda^*)} = J^1 \lambda^{21} \mathcal{M}^{1\lambda^* \lambda^* 2\lambda^*}.$$

This proves 34.1) $\rightarrow$ 34.2).

34.2) $\rightarrow$ 34.3): That $\lambda^{21} T^{21}$ is obvious. By IV A, T III, we secure $\mathcal{M}^{2\lambda^*} = \mathcal{M}^{2\kappa^*}$. $\mathcal{M}^{1\lambda^*} = \mathcal{M}^{1\kappa^*}$. Furthermore, $\lambda^{21} M^{2\lambda^* 1\lambda^*}$, and hence is $M^{2\kappa^* 1\lambda^*}$. Therefore, by III A, T II, $J^{1\lambda} \lambda^{21} u_{\alpha(\lambda^* \lambda)}^1 M^{2\kappa^*}$ for every $u_{\alpha(\lambda^* \lambda)}^1$. By 26.1), 26.2), and 26.6), we have

$$\epsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*} \cdot \text{comp cols } \mathcal{M}^{1\kappa^*} \cdot \lambda^{21} = J^{1\kappa^*} K^{*21} \epsilon_{\kappa}^1.$$

Thus by 27.21) and 27.32),

$$u_{\alpha(\lambda^* \lambda)}^1 \cdot \exists J^{1\kappa^*} \epsilon_{\kappa}^1 \ni J^{1\lambda} \lambda^{21} u_{\alpha(\lambda^* \lambda)}^1 = J^{1\lambda} (J^{1\kappa^*} K^{*21} \epsilon_{\kappa}^1) u_{\alpha(\lambda^* \lambda)}^1 \\ = J^{1\kappa^*} K^{*21} (J^{1\lambda} \epsilon_{\kappa}^1 u_{\alpha(\lambda^* \lambda)}^1) = J^{1\kappa^*} K^{*21} u_{\alpha(\lambda^* \lambda)}^1 \epsilon_{\kappa}^1.$$

Now by 22.51) and hypothesis,

$$J^2 \lambda^{*12} \mathcal{M}^{2\lambda^* \lambda^* 1\lambda^*} = \mathcal{M}'_{\alpha(\lambda^* \lambda)} = \mathcal{M}'_{\alpha(\lambda^* \lambda)} < \mathcal{M}^{1\kappa^* \kappa^* 2\kappa^*}.$$

Summarizing, we have

$$(a) \quad \epsilon_{\kappa}^1 \text{ cols } M^{1\kappa^*} \cdot \text{comp cols } \mathcal{M}^{1\kappa^*} \cdot \mathcal{M}'_{\alpha(\lambda^* \lambda)} < \mathcal{M}^{1\kappa^* \kappa^* 2\kappa^*}$$

Thus by 28.33), we have $\mathcal{M}^{1\kappa^*} \epsilon_{\kappa}^1 < \mathcal{M}^{1\kappa^* \kappa^* 2\kappa^*}$, from which we conclude that

$$\exists u_{\alpha(\lambda^* \lambda)}^1 \ni J^{1\lambda} \lambda^{21} u_{\alpha(\lambda^* \lambda)}^1 = J^{1\kappa^*} K^{*21} u_{\alpha(\lambda^* \lambda)}^1 M^{2\kappa^*}.$$

It follows that $J^{1\lambda} \lambda^{21} \mathcal{M}'_{\alpha(\lambda^* \lambda)} < \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa^*)}$, since $\mathcal{M}^{1\lambda^*} = \mathcal{M}^{1\kappa^*}$. Now

from (a), and $\lambda^{s1} \overline{\text{rows}} M^{1\kappa}$, we find by 28.33), 27.27) that $\varepsilon_{\kappa*}^s$ is by $\overline{\text{rows}} M^{s\kappa}$. Hence by 28.34), we have $J^1 \lambda^{s1} K^{1s} = \varepsilon_{\kappa*}^s$. From this relation, we can apply the same reasoning as above to obtain the last condition in 34.3).

34.3) $\rightarrow$ 0): By 26.1), 26.2), 26.4), and 26.6), we obtain $\varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \text{comp cols } \mathcal{M}^{1\kappa*}$. $\lambda^{s1} = J^{1\kappa*} K^{s1} \varepsilon_{\kappa}^1$, which together with $\mathcal{M}_{\cap(2\kappa* 2\kappa)}^2 \supset J^{1\kappa} \lambda^{s1} \mathcal{M}_{\cap(1\kappa* 1\kappa)}^1 = J^{1\kappa} \lambda^{s1} \mathcal{M}_{\cap(1\kappa 1\kappa)}^1$ gives, by 28.33), (2) $\sim$ (4), $\mathcal{M}_{\cap(1\kappa 1\kappa)}^1 \subset \mathcal{M}^{1\kappa \kappa* 2\kappa*}$. Again, by 28.33), (4) $\sim$ (1), and 27.27), we find that $\varepsilon_{\kappa*}^s \overline{\text{rows}} M^{s\kappa}$. Hence by 28.34), we obtain the relation $J^1 \lambda^{s1} K^{1s} = \varepsilon_{\kappa*}^s$. Now we can apply the same reasoning as above to obtain $\mathcal{M}_{\cap(2\kappa* 2\kappa)}^2 \subset \mathcal{M}^{2\kappa* \kappa 1\kappa}$. Thus 34.3) $\rightarrow$ 0) is proved.

That 0) $\rightarrow$ 34.5) follows from 32.1), 32.23), and 32.27).

34.5) $\rightarrow$ 34.6): From $\mathcal{M}^{2\kappa} = \mathcal{M}^{2\kappa*}$, $\mathcal{M}^{1\kappa*} = \mathcal{M}^{1\kappa}$, and IV A theory, we secure $\lambda^{s1} T^{s\kappa* 1\kappa} \cdot \text{comp cols } \mathcal{M}^{2\kappa*} \cdot \text{comp } \overline{\text{rows}} \mathcal{M}^{1\kappa}$. Further, from the relation $J^s K^{1s} \lambda^{s1} = \varepsilon_{\kappa}^1$, it follows that λ^{s1} is by cols $\mathcal{M}^{2\kappa* \kappa 1\kappa}$. By IV A, $\lambda^{s1} = J^1 \lambda^{s1} \varepsilon_{\kappa}^1 = J^1 \lambda^{s1} (J^s K^{1s} \lambda^{s1})$. Hence we have 34.6).

34.6) $\rightarrow$ 34.4): We note that $J^1 \lambda^{s1} (\varepsilon_{\kappa}^1 - J^s K^{1s} \lambda^{s1}) = O^{11}$. Now $\lambda^{s1} \text{ cols } \mathcal{M}^{2\kappa* \kappa 1\kappa}$, and hence $J^s K^{1s} \lambda^{s1} \text{ cols } M^{1\kappa}$. Thus, $(\varepsilon_{\kappa}^1 - J^s K^{1s} \lambda^{s1}) \text{ cols } M^{1\kappa}$. By hypothesis, $\lambda^{s1} \text{ comp } \overline{\text{rows}} \mathcal{M}^{1\kappa}$, and hence $J^s K^{1s} \lambda^{s1} = \varepsilon_{\kappa}^1$. Also, $\lambda^{s1} = J^1 \lambda^{s1} J^s K^{1s} \lambda^{s1} = J^s (J^1 \lambda^{s1} K^{1s}) \lambda^{s1}$, for $K^{1s} I^{1\kappa s\kappa*}$ and λ^{s1} type $\mathcal{M}^{2\kappa* \kappa 1\kappa} \overline{\mathcal{M}^{1\kappa \kappa* 2\kappa*}}$. Thus, we have $J^s (\varepsilon_{\kappa*}^s - J^1 \lambda^{s1} K^{1s}) \lambda^{s1} = O^{s1}$. Using similar reasoning as above, we secure $J^1 \lambda^{s1} K^{1s} = \varepsilon_{\kappa*}^s$. By 26.4), we now have the balance of 34.4).

0) $\rightarrow$ 34.7) follows from 31.131), 32.25), and 32.26).

34.7) $\rightarrow$ 34.1): From the first condition of 34.7), we obtain $\lambda^{s1} \text{ cols } \mathcal{M}_{\cap(2\kappa* 2\kappa)}^2$. Since λ^{s1} is by cols $\mathcal{M}^{2\kappa* \kappa 1\kappa}$, it follows from 22.51) that $J^s K^{1s} \lambda^{s1} \text{ cols } \mathcal{M}_{\cap(1\kappa 1\kappa)}^1$, whence, by hypothesis,

$J^1 \lambda^{21} (J^2 K^{12} \lambda^{21}) = \lambda^{21}$. Consequently, $J^1 \lambda^{21} (J^2 K^{12} \lambda^{21} - \epsilon_\kappa^1) = 0^{21}$. Since $(J^2 K^{12} \lambda^{21} - \epsilon_\kappa^1)$ cols $M^{1\kappa}$ and λ^{21} comp $\overline{\text{rows}} \mathcal{M}^{1\kappa}$, it follows that $J^2 K^{12} \lambda^{21} = \epsilon_\kappa^1$.

That $0) \rightarrow 34.8)$ follows from 32.27), 32.23), 32.1), and 32.26).

For $34.8) \rightarrow 34.6)$, we find that $\mathcal{M}^{2\lambda} = \mathcal{M}^{2\kappa*}$ gives $\epsilon_\lambda^2 = \epsilon_{\kappa*}^2$. Thus by IV A theory, λ^{21} comp cols $\mathcal{M}^{2\kappa*}$. Similarly, λ^{21} is complete by $\overline{\text{rows}} \mathcal{M}^{1\kappa}$. To secure the relation $\lambda^{21} = J^1 \lambda^{21} J^2 K^{12} \lambda^{21}$, we use the same reasoning as in the proof of 34.7) $\rightarrow$ 34.1).

§ 10

Theorem 35) is devoted to the study of spaces associated with K^{12} and its reciprocal.

T 35) $\lambda^{21} = \text{rec } K^{12} \quad :)$:

$$35.11) \mathcal{M}_{n(1\lambda^* 1\lambda)}^1 \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mathcal{M}_{n(2\kappa^* 2\kappa)}^2$$

$$35.12) \mathcal{M}_{n(2\lambda 2\lambda^*)}^2 \left(\begin{smallmatrix} 2 & \kappa \\ 1 & \lambda \end{smallmatrix} \right) \mathcal{M}_{n(1\kappa 1\kappa^*)}^1$$

$$35.21) \mathcal{M}_{n(2\kappa^* 2\kappa)}^2 \left(\begin{smallmatrix} 2 & \kappa \\ 1 & \lambda \end{smallmatrix} \right) \mathcal{M}_{n(1\lambda^* 1\lambda)}^1 < \mathcal{M}^{1\kappa^*} \sim \mathcal{M}_{n(1\lambda^* 1\lambda)}^1 = \mathcal{M}^{1\kappa^* \kappa^* 2\kappa}$$

$$\cdot) \cdot \mathcal{M}_{n(1\lambda^* 1\lambda)}^1 < \mathcal{M}_{n(1\kappa 1\kappa^*)}^1 \cdot \mathcal{M}_{n(2\kappa^* 2\kappa)}^2 < \mathcal{M}_{n(2\lambda 2\lambda^*)}^2$$

$$35.22) \mathcal{M}_{n(1\kappa 1\kappa^*)}^1 \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mathcal{M}_{n(2\lambda 2\lambda^*)}^2 < \mathcal{M}^{2\kappa} \sim \mathcal{M}_{n(2\lambda 2\lambda^*)}^2 = \mathcal{M}^{2\kappa \kappa 1\kappa^*}$$

$$\cdot) \cdot \mathcal{M}_{n(2\lambda 2\lambda^*)}^2 < \mathcal{M}_{n(2\kappa^* 2\kappa)}^2 \cdot \mathcal{M}_{n(1\kappa 1\kappa^*)}^1 < \mathcal{M}_{n(1\lambda^* 1\lambda)}^1$$

$$35.3) (1) \mathcal{M}^{1\kappa^*} > \mathcal{M}_{n(1\lambda^* 1\lambda)}^1 \left(\begin{smallmatrix} 1 & \kappa \\ 2 & \lambda \end{smallmatrix} \right) \mathcal{M}_{n(2\kappa^* 2\kappa)}^2 \cdot \mathcal{M}^{2\kappa} > \mathcal{M}_{n(2\lambda 2\lambda^*)}^2 \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mathcal{M}_{n(1\kappa 1\kappa^*)}^1$$

$$\sim (2) \mathcal{M}_{n(1\kappa 1\kappa^*)}^1 = \mathcal{M}_{n(1\lambda^* 1\lambda)}^1 = \mathcal{M}^{1\kappa^* \kappa^* 2\kappa}$$

$$\sim (3) \mathcal{M}_{n(2\kappa^* 2\kappa)}^2 = \mathcal{M}_{n(2\lambda 2\lambda^*)}^2 = \mathcal{M}^{2\kappa \kappa 1\kappa^*}$$

$$\sim (4) \mathcal{M}'_{n(\lambda^* \lambda)} = \mathcal{M}^{1_{K^*} 2_K} \cdot \mathcal{M}^2_{n(z_\lambda z_{\lambda^*})} = \mathcal{M}^{2_K 1_{K^*}}$$

$$35.41) K^{*21} = \lambda^{21} \cdot) .$$

$$35.411) K^{12} M^{12} \cdot M^{1_{K^*} 2_K} \cdot M^{12} K^{12} = M^{1_{K^*} 2_K} K^{12} = \begin{cases} 1 & (K^{12} \neq 0^{12}) \\ 0 & (K^{12} = 0^{12}) \end{cases}$$

$$35.412) \mathcal{M}^{1_K 1_{K^*} 2_K} = \mathcal{M}^{1_K} = \mathcal{M}^{1_{K^*}} = \mathcal{M}'_{n(1_K 1_{K^*})} = \mathcal{M}^{1_{K^*} 1_K} \\ = \mathcal{M}^{1_\lambda 1_{\lambda^*}} = \mathcal{M}^{1_\lambda 1_{\lambda^*}} = \mathcal{M}'_{n(1_\lambda 1_{\lambda^*})} = \mathcal{M}^{1_{\lambda^*}} = \mathcal{M}^{1_\lambda} .$$

$$35.42) (1) \mathcal{M}^{1_{K^*}} \supset \mathcal{M}'_{n(1_{\lambda^*} 1_\lambda)} \left(\begin{smallmatrix} 1 & \lambda \\ 2 & K \end{smallmatrix} \right) \mathcal{M}^2_{n(2_{K^*} 2_K)} \\ \cdot \left(\begin{smallmatrix} 1 & K^* \\ 2 & \lambda^* \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & \lambda \\ 2 & K \end{smallmatrix} \right) \text{ on } (\mathcal{M}'_{n(1_{\lambda^*} 1_\lambda)} \cdot \mathcal{M}^2_{n(2_{K^*} 2_K)})$$

$$\sim (2) \mathcal{M}^{2_K} \supset \mathcal{M}^2_{n(2_\lambda 2_{\lambda^*})} \left(\begin{smallmatrix} 2 & \lambda^* \\ 1 & K^* \end{smallmatrix} \right) \mathcal{M}'_{n(1_K 1_{K^*})} \\ \cdot \left(\begin{smallmatrix} 2 & K \\ 1 & \lambda \end{smallmatrix} \right) = \left(\begin{smallmatrix} 2 & \lambda^* \\ 1 & K^* \end{smallmatrix} \right) \text{ on } (\mathcal{M}^2_{n(2_\lambda 2_{\lambda^*})} \cdot \mathcal{M}'_{n(1_K 1_{K^*})})$$

$$\sim (3) K^{*21} = \lambda^{21} \sim (4) e^1_{K^*} = e^1_K \sim (5) e^2_K = e^2_{K^*} .$$

$$35.5) \mu^1_{n(1_{\lambda^*} 1_\lambda)} \left(\begin{smallmatrix} 1 & K^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mu^2_{n(2_{K^*} 2_K)} \cdot \mu^1_{n(1_K 1_{K^*})} \left(\begin{smallmatrix} 1 & \lambda \\ 2 & K \end{smallmatrix} \right) \mu^2_{n(2_\lambda 2_{\lambda^*})} \cdot) .$$

$$J^1 \overline{\mu^1_{n(1_{\lambda^*} 1_\lambda)} \mu^1_{n(1_K 1_{K^*})}} = J^2 \overline{\mu^2_{n(2_{K^*} 2_K)} \mu^2_{n(2_\lambda 2_{\lambda^*})}}$$

$$35.61) K^{12} M^{1_{K^*} 2_K} \sim \lambda^{21} M^{21}$$

$$35.62) \lambda^{21} M^{2_{\lambda^*} 1_\lambda} \sim K^{12} M^{12}$$

In 5.7), let $\mathcal{M}'_f$ and $\mathcal{M}^{2_f}$ ($f = 1, 2$) be corresponding subsets of $\mathcal{M}'_{n(1_{\lambda^*} 1_\lambda)}$ and $\mathcal{M}^2_{n(2_{K^*} 2_K)}$ respectively in the one-to-one correspondence of 35.11) and similarly, let $\mathcal{M}'_f$ and $\mathcal{M}^{2_f}$ ($f = 1, 2$) be corresponding subsets in the one-to-one correspondence of 35.12). We state the following theorems:

$$35.71) \mathcal{M}'_2 O' \mathcal{M}'_1 \sim \mathcal{M}^{2_2} O^2 \mathcal{M}^{2_1}$$

$$35.72) \mathcal{M}^{1_K} = n(\mathcal{M}'_{n(1_K 1_{K^*})} \cdot O' \mathcal{M}'_1) \sim \mathcal{M}^{2_\lambda} = n(\mathcal{M}^2_{n(2_\lambda 2_{\lambda^*})} \cdot O^2 \mathcal{M}^{2_{K^*}})$$

$$: \mathcal{M}_1^{\lambda*} = \cap (\mathcal{M}_{\cap(\lambda^* \lambda)}^{\lambda} \cdot O' \mathcal{M}_2^{\lambda}) \sim \mathcal{M}_1^{2\kappa*} = \cap (\mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2 \cdot O^2 \mathcal{M}_2^{2\lambda})$$

$$35.73) \left(U(\mathcal{M}_1^{\lambda*} \cdot \mathcal{M}_2^{\lambda*}) \right)_L \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \left(U(\mathcal{M}_1^{2\kappa*} \cdot \mathcal{M}_2^{2\kappa*}) \right)_L$$

By 22.55) and 32.25) we secure $\mathcal{M}_{\cap(\lambda^* \lambda)}^{\lambda} \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2$.
Further, by 32.26) and IV A, T II, we have

$$J^2 \lambda^{*12} = J^{2\kappa} K^{12} \quad \text{on} \quad \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2$$

$$J^{1\kappa} K^{*21} = J^{1\kappa} K^{*21} \quad \text{on} \quad \mathcal{M}_{\cap(\lambda^* \lambda)}^{\lambda}$$

from which, we secure 35.11). By parity, we obtain 35.12).

35.22) follows from 32.25), 32.26), and 24.5), (1) $\sim$ (3).
35.21) is the hermitian parity of 35.22).

In 35.3), we observe that (1) $\sim$ (2) $\cdot$ (3) $\sim$ (4) follows
from 35.21) and 35.22). From 32.25) and 24.6), we secure immediately the equivalence between (2) and (3).

If $K^{*21} = \lambda^{21}$, then $\varepsilon_{\kappa^*}^1 \equiv J^2 K^{12} K^{*21} = \varepsilon_{\kappa}^1$. It follows
that $K^{12} M^{12} \cdot M^{1\kappa^* 2\kappa}$. By 21.2), we secure the values of moduli.
Part 35.412) is evident from 35.411).

To prove 35.42), we use the following scheme: (4) $\rightarrow$ (3) $\rightarrow$
(1) $\rightarrow$ (4). We shall make use of 34.1) in the proof of (4) $\rightarrow$ (3).
Evidently, $K^{*21} T^{2\kappa^* 1\kappa} \cdot \text{comp cols } \mathcal{M}^{2\kappa^*} \cdot \text{comp rows } \mathcal{M}^{1\kappa}$, and also
 $J^2 K^{12} K^{*21} = \varepsilon_{\kappa^*}^1 = \varepsilon_{\kappa}^1$. Since $K^{12} M^{1\kappa^* 2}$, it follows from the hypothesis that K^{12} is $M^{1\kappa^* 2}$ and hence M^{12} . Thus by 22.1),

$$\mathcal{M}^{1\kappa} = \mathcal{M}^{1\kappa^* K^{*2\kappa^*}} \cdot \mathcal{M}^{2\kappa^*} = \mathcal{M}^{2\kappa^* K^{1\kappa}}$$

Therefore, we have

$$\mathcal{M}_{\cap(\lambda^* \lambda)}^{\lambda} \subset \mathcal{M}^{1\kappa^* K^{*2\kappa^*}} \cdot \mathcal{M}_{\cap(2\kappa^* 2\kappa)}^2 \subset \mathcal{M}^{2\kappa^* K^{1\kappa}}$$

Thus 34.1) is satisfied and $K^{*21} = \text{rec } K^{12}$.

If $K^{*21} = \lambda^{21}$, then by 35.41), $\mathcal{M}^{1\kappa^* K^{*2\kappa^*}} = \mathcal{M}^{1\kappa^* K^{*2\kappa}}$. Thus

by 35.21), the first part of (1) is proved, and it follows at once that $(\begin{smallmatrix} 1 & \lambda \\ 2 & \kappa \end{smallmatrix}) = (\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix})$. Further by 32.29), it is obvious that $(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix}) = (\begin{smallmatrix} 1 & \lambda^* \\ 2 & \kappa \end{smallmatrix})$. Thus the second part of (1) is also established.

For (1) $\rightarrow$ (4), we note that K^{12} cols $\mathcal{M}_{n(2\kappa^* 2\kappa)}^2$. By hypothesis, $J^2 \lambda^{*12} = J^2 K^{12}$ on $\mathcal{M}_{n(2\kappa^* 2\kappa)}^2$. Hence $\varepsilon_{\kappa}^1 = J^2 \lambda^{*12} K^{*21} = J^2 K^{12} K^{*21} \equiv \varepsilon_{\kappa^*}^1$. By parity, we secure the balance of 35.42).

To prove 35.5), we note, from the hypothesis, that

$$\mu_{n(\lambda^* \lambda)}^1 = J^2 \lambda^{*12} \mu_{n(2\kappa^* 2\kappa)}^2 \quad \cdot \quad \mu_{n(\lambda' \kappa^*)}^1 = J^2 K^{12} \mu_{n(2\lambda^* 2\lambda)}^2.$$

Hence

$$J^1 \overline{\mu_{n(\lambda^* \lambda)}^1} \mu_{n(\lambda' \kappa^*)}^1 = J^1 (J^2 \overline{\mu_{n(2\kappa^* 2\kappa)}^2} \lambda^{21}) \mu_{n(\lambda' \kappa^*)}^1,$$

which, by virtue of 32.27) and 32.24), is equal to

$$J^2 \overline{\mu_{n(2\kappa^* 2\kappa)}^2} (J^1 \lambda^{21} \mu_{n(\lambda' \kappa^*)}^1).$$

But, since $K^{12} \mathbb{I}^{1\kappa^* 2\kappa^*}$,

$$\begin{aligned} J^1 \lambda^{21} \mu_{n(\lambda' \kappa^*)}^1 &= J^1 \lambda^{21} J^2 K^{12} \mu_{n(2\lambda^* 2\lambda)}^2 = J^2 (J^1 \lambda^{21} K^{12}) \mu_{n(2\lambda^* 2\lambda)}^2 \\ &= J^2 \varepsilon_{\kappa^*}^2 \mu_{n(2\lambda^* 2\lambda)}^2 = \mu_{n(2\lambda^* 2\lambda)}^2, \end{aligned}$$

by virtue of 32.1), 32.25), and 32.23). Thus 35.5) is proved.

If $K^{12} M^{1\kappa^* 2\kappa^*}$, then $\lambda^{21} = J^{1\kappa^*} J^{2\kappa} K^{*21} K^{12} K^{*21}$, from which $\lambda^{21} M^{21}$ follows immediately. Conversely, if $\lambda^{21} M^{21}$, then from $J^2 K^{12} \lambda^{21} = \varepsilon_{\kappa}^1$, we find that $\varepsilon_{\kappa}^1 M^{1\kappa^* 21}$ and hence by 21.6), K^{12} is $M^{1\kappa^* 2\kappa^*}$.

To prove 35.62), we note from 32.29) that $K^{12} = \text{rec } \lambda^{21}$.

Now applying 35.61) just proved, we secure the result.

35.71) follows from 35.5).

35.72₁): Let us first prove the condition is necessary.

By hypothesis, we have $\mathcal{M}_2^{\lambda'} O' \mathcal{M}^{\lambda^*} \subset \mathcal{M}_{n(\lambda' \kappa^*)}^1$. Then by 35.71),

$\mathfrak{M}_2^{2\lambda} \circ^2 \mathfrak{M}_1^{2\kappa*}$. It follows that

$$\mathfrak{M}_2^{2\lambda} \subset \cap (\mathfrak{M}_{\cap(z_\lambda z_{\lambda*})}^{2\lambda} \cdot \circ^2 \mathfrak{M}_1^{2\kappa*}).$$

On the other hand, let $\mathfrak{M}_0^{2\lambda} \equiv \cap (\mathfrak{M}_{\cap(z_\lambda z_{\lambda*})}^{2\lambda} \cdot \circ^2 \mathfrak{M}_1^{2\kappa*})$, and let

$$\mathfrak{M}_0^{1\kappa} \subset \mathfrak{M}_{\cap(1_\kappa 1_{\kappa*})}^{1\kappa} \quad \ni \quad \mathfrak{M}_0^{1\kappa} \left(\frac{1}{2} \lambda \right) \mathfrak{M}_0^{2\lambda}.$$

Then by 35.71), $\mathfrak{M}_0^{1\kappa} \circ^1 \mathfrak{M}_1^{1\lambda*}$, and hence by hypothesis, $\mathfrak{M}_0^{1\kappa} \subset \mathfrak{M}_2^{1\kappa}$.

Thus

$$\mathfrak{M}_0^{2\lambda} = J^1 \lambda^{21} \mathfrak{M}_0^{1\kappa} \subset J^1 \lambda^{21} \mathfrak{M}_2^{1\kappa} = \mathfrak{M}_2^{2\lambda}.$$

The necessary condition is now proved.

By 32.29) $K^{12} = \text{rec } \lambda^{21}$. From this, the sufficiency is secured by mutual parity.

Again, by 32.29), $\lambda^{*12} = \text{rec } K^{*21}$. Thus 35.72₂) is secured from the first part by hermitian parity.

35.73): From the hypothesis, it is evident that

$$(\cup(\mathfrak{M}_1^{2\kappa*} \cdot \mathfrak{M}_2^{2\kappa*}))_L \subset \mathfrak{M}_{\cap(z_{\kappa*} z_{\kappa})}^{2\kappa} \quad \text{and} \quad (\cup(\mathfrak{M}_1^{1\lambda*} \cdot \mathfrak{M}_2^{1\lambda*}))_L \subset \mathfrak{M}_{\cap(1_{\lambda*} 1_{\lambda})}^{1\lambda}.$$

One can prove readily the relation

$$J^1 K^{*21} (\cup(\mathfrak{M}_1^{1\lambda*} \cdot \mathfrak{M}_2^{1\lambda*}))_L = (\cup(\mathfrak{M}_1^{2\kappa*} \cdot \mathfrak{M}_2^{2\kappa*}))_L$$

from which 35.73) follows by 35.11).

In the study of the composition theorem in IV C, use is made of the J-linear closure property, LJ, quite extensively. However, the notion LJ which is indispensable in the study of modular matrices loses its power in the non-modular case. The writer believes that a new notion is necessary for the study of the composition theorem in the present situation and for further development in the theory of non-modular matrices.

In addition to the four types of modularity that are always present in ϵ_{κ}^1 , as we have already studied in 21.1), we find that ϵ_{κ}^1 is also modular relative to $(\epsilon_{\kappa}^1, \epsilon_{\lambda}^1)$, and $M^{1\kappa*1\lambda}\epsilon_{\kappa}^1 = 1$ or 0 according as $K^{1\kappa} \neq 0^{1\kappa}$ or $K^{1\kappa} = 0^{1\kappa}$. From 32.23), it is evident that ϵ_{κ}^1 is self-reciprocal.

IV. MODULAR RECIPROCALS OF NON-MODULAR MATRICES

In this section, we shall strengthen the conditions in the definition of the reciprocal. On account of the stronger conditions imposed upon the reciprocal, the mutuality property is not preserved. On the other hand, some new properties are obtained. Whenever the reciprocal λ^{s1} of K^{1s} exists according to the definition in this section, it is also a reciprocal according to the definition in Chapter III, but not conversely. Furthermore, we bring out an interesting fact that, when λ^{s1} is modular relative to $(\epsilon^s \ \epsilon^1)$, K^{1s} must be modular relative to $(\epsilon_{\kappa^*}^1 \ \epsilon_{\kappa}^s)$, and conversely.

The basis for the present section is

$$U^D \cdot \beta^1 \cdot \beta^2 \cdot \epsilon^{1PH} \cdot \epsilon^{2PH} \cdot K^{12} T^{1s}$$

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$$\begin{aligned} D \ 41.1r \quad \lambda^{s1} = R \text{ Rec } K^{1s} \cdot \equiv \cdot \lambda^{s1} T^{s1} \cdot J^s K^{1s} \lambda^{s1} = \epsilon_{\kappa}^1 \\ \cdot J^1 \lambda^{s1} m^1 < m^{2\kappa^*} \end{aligned}$$

$$\begin{aligned} D \ 41.1\bar{r} \quad \lambda^{s1} = L \text{ Rec } K^{1s} \cdot \equiv \cdot \lambda^{s1} T^{s1} \cdot J^1 \lambda^{s1} K^{1s} = \epsilon_{\kappa^*}^s \\ \cdot J^s \lambda^{*1s} m^2 < m^{1\kappa} \end{aligned}$$

$$D \ 41.2) \quad \lambda^{s1} = \text{Rec } K^{1s} \cdot \equiv \cdot \lambda^{s1} R \text{ Rec } K^{1s} \cdot L \text{ Rec } K^{1s}$$

From the definition of right reciprocal ($R \text{ Rec } K^{1s}$), we state the following theorem:

$$T \ 41.11) \quad \exists \lambda^{s1} = R \operatorname{Rec} K^{1s} \cdot) \cdot \exists! \lambda^{s1}$$

$$41.12) \quad \lambda^{s1} = R \operatorname{Rec} K^{1s} \cdot \sim \cdot \lambda^{s1} M^{s\kappa*1} \cdot J^s K^{1s} \lambda^{s1} = \varepsilon_{\kappa}^1 \\ \cdot) \cdot \lambda^{s1} M^{s1} \cdot \operatorname{cols} \mathcal{M}^{2\kappa*\kappa'_{\kappa}}$$

$$41.13) \quad \exists \lambda^{s1} = R \operatorname{Rec} K^{1s} \cdot \sim \cdot K^{1s} M^{1\kappa*s\kappa} \cdot \sim \cdot \mathcal{M}'_{\kappa} \subset \mathcal{M}^{1\kappa*} \\ \cdot \sim \cdot \mathcal{M}'_{\kappa} \cdot 2\kappa* = \mathcal{M}'_{\kappa} \cdot) \cdot K^{1s} I^{1\kappa*s\kappa} \cdot J^{1\kappa*} K^{*s1} \varepsilon_{\kappa}^1 = R \operatorname{Rec} K^{1s}$$

$$41.14) \quad \lambda^{s1} = R \operatorname{Rec} K^{1s} \cdot) \cdot \lambda^{s1} \operatorname{cols} M^{s\kappa*} \cdot \operatorname{comp} \operatorname{cols} \mathcal{M}^{2\kappa*} \\ \cdot J^s K^{1s} \lambda^{s1} = \varepsilon_{\kappa}^1 \cdot \mathcal{M}'_{\kappa} \subset \mathcal{M}^{1\kappa*} \cdot 2\kappa*$$

$$41.2) \quad \exists \lambda_L^{s1} \cdot \exists \lambda_R^{s1} \cdot) \cdot \exists! \lambda_L^{s1} = \exists! \lambda_R^{s1} = \exists! \lambda^{s1}$$

From the last condition in D 41.1r, $\lambda^{s1} \operatorname{cols} M^{s\kappa*}$. Suppose there exist two right reciprocals λ_{R1}^{s1} and λ_{R2}^{s1} . Then $J^s K^{1s} (\lambda_{R1}^{s1} - \lambda_{R2}^{s1}) = 0^{11}$. Since $K^{1s} \operatorname{comp} \overline{\operatorname{rows}} \mathcal{M}^{2\kappa*}$, the uniqueness follows.

To prove 41.12), we note that $\lambda^{s1} \overline{\operatorname{rows}} M^1 \cdot J^1 \lambda^{s1} \mathcal{M}' \subset \mathcal{M}^{2\kappa*}$ give $\lambda^{s1} M^{s\kappa*1}$ by the corollary to Hellinger-Toeplitz theorem. The sufficiency is obvious. Finally, from the second and third conditions in D 41.1r, we see that $\lambda^{s1} \operatorname{cols} \mathcal{M}^{2\kappa*\kappa'_{\kappa}}$.

To prove 41.13), we note that $J^s K^{1s} \lambda^{s1} = \varepsilon_{\kappa}^1$. Since K^{1s} is $M^{1\kappa*s\kappa}$ and λ^{s1} is M^{s1} , we have $\varepsilon_{\kappa}^1 M^{1\kappa*s1}$, which by 21.6) gives $K^{1s} M^{1\kappa*s\kappa}$. Conversely, if $K^{1s} M^{1\kappa*s\kappa}$, then $\varepsilon_{\kappa}^1 M^{1\kappa*s1}$. Define $\lambda^{s1} = J^{1\kappa*} K^{*s1} \varepsilon_{\kappa}^1$. Then $\lambda^{s1} M^{s\kappa*1}$, since $K^{1s} M^{1\kappa*s\kappa}$. Moreover, $J^s K^{1s} (J^{1\kappa*} K^{*s1} \varepsilon_{\kappa}^1) = J^{1\kappa*} (J^s K^{1s} K^{*s1}) \varepsilon_{\kappa}^1 = \varepsilon_{\kappa}^1$. Thus the conditions in 41.12) are satisfied. This proves the first equivalence. The second equivalence follows from 21.6) and the third, from 22.51). Finally, from $K^{1s} M^{1\kappa*s\kappa}$ we secure the $I^{1\kappa*s\kappa}$ -property of K^{1s} . Hence by 25.3), we have $K^{1s} I^{1\kappa*s\kappa}$.

In 41.14), the first and third conditions are evident. Since $\lambda^{21} M^{21}$ and $J^2 K^{12} \lambda^{21} = \epsilon_\kappa^1$, then $\epsilon_\kappa^1 M^{1\kappa*1}$. Thus by 21.6), $K^{12} M^{1\kappa*2\kappa} \cdot \epsilon_\kappa^1 M^{1\kappa*1\kappa}$ and hence by 27.1), $m^{1\kappa* \kappa* 2\kappa} = m^{1\kappa* \epsilon_\kappa^1 \cdot 1\kappa}$. But $\lambda^{21} = J^{1\kappa*} K^{*21} \epsilon_\kappa^1$, from which the second and fourth conditions are obtained by 28.33).

Properties of a reciprocal λ^{21} of K^{12} are stated in the following theorem:

T 42.1) $\lambda^{21} = \text{Rec } K^{12} \quad .)$.

$$42.11) \quad \lambda^{21} M^{2\lambda^* 1\lambda^*} \cdot M^{21} \cdot I^{21}$$

42.12) λ^{21} is also the reciprocal of K^{12} according to the definition in Chapter III.

$$42.13) \quad m^{1\kappa*} > m^{1\kappa} = m^{1\kappa*} > m^{1\lambda} = J^2 \lambda^{*12} m^{2\lambda}.$$

$$42.14) \quad m^{1\kappa \cdot 2\kappa*} = m^{1\lambda*} \cdot m^{1\kappa \kappa* 2\kappa*} = m^{1\lambda}$$

$$42.15) \quad J^{1\kappa*} K^{*21} = J^1 \lambda^{21} \quad \text{on } m^{1\kappa}$$

$$42.16) \quad m^{1\lambda \lambda^* 2\lambda^*} = J^2 \lambda^{*12} m^{2\lambda*} \cdot m_{\lambda}^{2\lambda* \cdot 1\lambda} = m^{2\lambda*}$$

$$42.2) \quad \exists \lambda^{21} = \text{Rec } K^{12} \quad .\sim. \quad K^{12} M^{1\kappa* 2\kappa}$$

$$.) \cdot J^{1\kappa*} J^{2\kappa} K^{*21} K^{12} K^{*21} = \text{Rec } K^{12}.$$

By 41.12), $\lambda^{21} M^{21}$ and hence $\lambda^{21} M^{2\lambda^* 1\lambda^*} \cdot I^{21}$. 42.12) follows from 41.14) and its hermitian parity for left reciprocal.

From 42.12) we obtain: 42.13) by 32.23), 21.42) and 21.5); 42.14) by 42.13), 32.25), and 41.13); 42.15) by 41.13) and 32.26); 42.16) by 24.1), 24.2), and 42.13); and finally, 42.2) by 41.13).

§ 12

As in the preceding chapter, we shall impose the condition T_S^{12} upon K^{12} . The basis for the study of classical reciprocals will be as follows:

$$\mathcal{U}^D \cdot \beta^1 \cdot \beta^2 \cdot \varepsilon^{PH} \cdot \varepsilon^{2PH} \cdot K^{12} T_S^{12}$$

$$D\ 42.1r \quad \lambda^{21} = R \operatorname{Rec} K^{12} \equiv \lambda^{21} M^{21} \cdot J^2 K^{12} \lambda^{21} = \varepsilon^1$$

$$D\ 42.1\chi \quad \lambda^{21} = L \operatorname{Rec} K^{12} \equiv \lambda^{21} M^{21} \cdot J^1 \lambda^{21} K^{12} = \varepsilon^2$$

$$D\ 42.2) \quad \lambda^{21} = \operatorname{Rec} K^{12} \equiv \lambda^{21} R \operatorname{Rec} K^{12} \cdot L \operatorname{Rec} K^{12}$$

From the preceding definitions, we obtain immediately the following corollaries:

$$CD\ 42.1r \quad \lambda^{21} = R \operatorname{Rec} K^{12} \quad .). \quad \lambda^{21} \operatorname{cols} \mathcal{M}^{2 \times 1}$$

$$CD\ 42.1\chi \quad \lambda^{21} = L \operatorname{Rec} K^{12} \quad .). \quad \lambda^{21} \overline{\operatorname{rows}} \mathcal{M}^{1 \times 2}$$

Theorem 43) states the properties of classical reciprocals.

$$T\ 43.11) \quad (1) \quad \mathbb{E} \lambda_{\circ R}^{21} \sim. \quad (2) \quad K^{12} M^{1 \times 2 \times 2 \times} \cdot \operatorname{comp} \operatorname{cols} \mathcal{M}^1 \sim.$$

$$(3) \quad \varepsilon^1 M^{1 \times 2 \times 1 \times} \cdot K^{12} I^{12} \cdot J^{1 \times 2} K^{2 \times 1} \varepsilon_{\kappa}^1 = R \operatorname{Rec} K^{12}$$

$$43.12) \quad (1) \quad \mathbb{E} \lambda_{\circ L}^{21} \sim. \quad (2) \quad K^{12} M^{1 \times 2 \times 2 \times} \cdot \operatorname{comp} \overline{\operatorname{rows}} \mathcal{M}^2 \sim.$$

$$(3) \quad \varepsilon^2 M^{2 \times 2 \times 2 \times} \cdot K^{12} I^{12} \cdot J^{2 \times} \varepsilon_{\kappa}^2 K^{2 \times 1} = L \operatorname{Rec} K^{12}$$

$$43.2) \quad \mathbb{E} \lambda_{\circ}^{21} \sim. \quad K^{12} \operatorname{comp} \operatorname{cols} \mathcal{M}^1 \cdot \operatorname{comp} \overline{\operatorname{rows}} \mathcal{M}^2 \cdot M^{1 \times 2 \times 2 \times} \sim.$$

$$\sim. \quad \varepsilon^1 M^{1 \times 2 \times 1 \times} \cdot \varepsilon^2 M^{2 \times 2 \times 2 \times}$$

$$\cdot K^{12} I^{12} \cdot J^{1 \times 2} J^{2 \times} K^{2 \times 1} K^{12} K^{2 \times 1} = \operatorname{Rec} K^{12}$$

$$43.3) \quad \mathbb{E} \lambda_{\circ L}^{21} \cdot \mathbb{E} \lambda_{\circ R}^{21} \cdot \mathbb{E} \lambda_{\circ L}^{21} = \mathbb{E} \lambda_{\circ R}^{21} = \mathbb{E} \lambda_{\circ}^{21}$$

43.12) is obtained from 43.11) by hermitian parity. 43.2) is the combined result of 43.11) and 43.12). 43.3) follows from CD 42.1r, CD 42.1X, and $K^{12} I^{12}$; and 43.4) follows from 43.11) and 43.12).

To prove 43.5), consider $J^1 \bar{u}^T K^{12} = O^s$. Since $\varepsilon^s \neq O^{ss}$, let $\varepsilon^s(p^s) \neq O^s$. Define $\varphi^{s1} \equiv \varepsilon^s(p^s) \bar{u}^T$; then, by III A, T IV, $\varphi^{s1} M^{s1}$ and $J^1 \varphi^{s1} K^{12} = O^{ss}$. Now $(\lambda_L^{s1} + \varphi^{s1})^{M^{s1}} \cdot J^1 (\lambda_L^{s1} + \varphi^{s1}) K^{12} = J^1 \lambda_L^{s1} K^{12} + J^1 \varphi^{s1} K^{12} = \varepsilon^s$; hence $\lambda_L^{s1} + \varphi^{s1}$ is effective as a left classical reciprocal of K^{12} . But, by hypothesis, the left classical reciprocal is unique; so $\lambda_L^{s1} + \varphi^{s1} = \lambda_L^{s1}$. Since $\varepsilon^s(p^s) \neq O^s$, $u^1 = O^1$. This proves that $K^{12} \text{ comp cols } \mathcal{M}'$.

In 43.6), the first two parts are obvious from the definitions and 43.2). For the third part, we find from 43.2), 43.6₂) 42.12), and 32.23) that $\mathcal{M}^{2\lambda} = \mathcal{M}^{2\kappa*}$ and $\mathcal{M}^{\lambda*} = \mathcal{M}^{\lambda\kappa}$. Thus

$$J^{\lambda\lambda^2} \mathcal{M}'_{\alpha(\lambda\kappa)} = J^{\lambda\lambda^2} \mathcal{M}'_{\alpha(\lambda\kappa)} = J^{\lambda\lambda^2} \mathcal{M}^{\lambda\kappa} = \mathcal{M}^{2\lambda} = \mathcal{M}^{2\kappa*} = \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)}.$$

Similarly, $J^{2\lambda^*} \lambda^{*12} \mathcal{M}^2_{\alpha(2\kappa^* 2\kappa)} = \mathcal{M}'_{\alpha(\lambda\kappa^* \lambda^* \kappa)}$. 43.6₃) is now proved.

Since $\lambda^{ss} M^{ss}$ and $\lambda^{s1} M^{s1}$, we find $\lambda^{s1} = J^s \lambda^{ss} \lambda^{s1} M^{s1}$; further, since $K^{ss} I^{ss}$, we have

$$\begin{aligned} J^s K^{1s} \lambda^{s1} &= J^s K^{1s} (J^s \lambda^{ss} \lambda^{s1}) = J^s (J^s K^{1s} \lambda^{ss}) \lambda^{s1} \\ &= J^s (J^s (J^s K^{12} K^{2s}) \lambda^{ss}) \lambda^{s1} = J^s (J^s K^{12} (J^s K^{2s} \lambda^{ss}) \lambda^{s1}. \end{aligned}$$

Thus $J^s K^{1s} \lambda^{s1} = \varepsilon^1$. Similarly, we have $J^1 \lambda^{s1} K^{12} = \varepsilon^s$.

The following theorem is evident:

$$\mathcal{M}' > \mathcal{M}'_0 > \mathcal{M}^{\lambda\kappa} \quad \cdot \quad \mathcal{M}^2 > \mathcal{M}^2_0 > \mathcal{M}^{2\kappa*} \quad :):$$

$$1) \lambda^{s1} = R \text{ Rec } K^{12} \sim \lambda^{s1} \text{ cols } M^s. J^s K^{12} \lambda^{s1} = \varepsilon^1. J^{\lambda\lambda^2} \mathcal{M}' < \mathcal{M}^2_0$$

$$2) \lambda^{s1} = L \text{ Rec } K^{12} \sim \lambda^{s1} \text{ rows } M^1. J^1 \lambda^{s1} K^{12} = \varepsilon^s. J^{2\lambda^* 2} \mathcal{M}^2 < \mathcal{M}'_0$$

§ 13

The following theorem is devoted to the study of (1) the modularities that are always present in K^{12} , λ^{21} , and the six positive, hermitian matrices

$$\varepsilon_{\kappa}^1 = \varepsilon_{\lambda^*}^1, \varepsilon_{\kappa^*}^1, \varepsilon_{\lambda}^1, \varepsilon_{\kappa^*}^2 = \varepsilon_{\lambda}^2, \varepsilon_{\kappa}^2, \varepsilon_{\lambda^*}^2;$$

(2) necessary and sufficient conditions for the modularities that are not always present in those matrices; and (3) the relation of the values of moduli of K^{12} , λ^{21} , and the six derived positive, hermitian matrices.

T 44) $\lambda^{21} = \text{Rec } K^{12} \quad ;):$

$$44.11) K^{12} M^{12\kappa} \cdot M^{1\kappa^*2} \cdot M^{1\kappa}2\kappa \cdot M^{1\kappa^*}2\kappa^* \cdot M^{1\kappa^*}2\kappa$$

$$44.12) \lambda^{21} 15 \text{ mods} \cdot M^{2\lambda^*1\lambda} \text{ in general}$$

$$44.13) \varepsilon_{\kappa}^1 (= \varepsilon_{\lambda^*}^1) \text{ is modular relative to } (\varepsilon^1 \varepsilon^1), (\varepsilon^1 \varepsilon_{\kappa}^1), \\ (\varepsilon_{\kappa}^1 \varepsilon^1), (\varepsilon_{\kappa}^1 \varepsilon_{\kappa}^1), (\varepsilon^1 \varepsilon_{\kappa^*}^1), (\varepsilon_{\kappa^*}^1 \varepsilon^1), (\varepsilon_{\kappa}^1 \varepsilon_{\kappa^*}^1), (\varepsilon_{\kappa^*}^1 \varepsilon_{\kappa}^1), \\ (\varepsilon_{\kappa^*}^1 \varepsilon_{\kappa^*}^1), (\varepsilon_{\kappa^*}^1 \varepsilon_{\lambda}^1), \text{ and } (\varepsilon_{\lambda}^1 \varepsilon_{\kappa^*}^1).$$

$$44.14) \varepsilon_{\lambda}^1 16 \text{ mods}$$

$$44.2) \quad (1) K^{12} M^{12}$$

.. (2) K^{12} any one of the remaining 11 mods

.. (3) $\varepsilon_{\kappa}^1 = \varepsilon_{\lambda^*}^1$ any one of the remaining 5 mods

.. (4) $\lambda^{21} M^{2\lambda^*1\lambda}$

.. (5) $\varepsilon_{\kappa^*}^1$ any one of the remaining 15 mods

.) (6) $K^{12} = J^{1\lambda} J^{2\lambda^*} \lambda^{*12} \lambda^{21} \lambda^{*12} \cdot K^{12} = \text{Rec } \lambda^{21}$

From T 41) when a matrix λ^{21} is a reciprocal of K^{12} , it is necessary that λ^{21} be M^{21} . Since K^{12} is not in general M^{12} , the mutuality property is not preserved. The only parity that we may use to deduce certain properties concerning K^{12} and its reciprocal λ^{21} is the hermitian parity in which $(1,2,K,\lambda)$ are replaced by $(2,1,K^*,\lambda^*)$ respectively.

From the first part of T 44), we notice that there are 76 moduli to be computed. (In IV C, there are 128 possible moduli. But in the present situation, 52 moduli are not, in general, finite.) Neglecting the trivial case when $K^{12} = O^{12}$, we see that the moduli whose values are 1 can be classified into two types: (1) those which are self-modular, and (2) those which are obtained by IV A applications. (See 44.31). To compute the values of the 76 moduli, we shall reduce the problem by making use of the following principles: First, by virtue of the theorem

$$2^D \cdot \mathfrak{P}^1 \cdot \mathfrak{P}^2 \cdot \mathfrak{P}^3 \cdot \epsilon^1 \cdot \epsilon^2 \cdot \kappa^{12} \text{ cols } M' \cdot \varphi^{13} M'^{13} \cdot M'^{13} \varphi^{13} = M'^{13} \varphi^{13}$$

we can discard the moduli involving ϵ_{κ}^1 , $\epsilon_{\kappa^*}^2$. Secondly, on account of the hermitian property of the six ϵ 's we may discard the moduli of their conjugate transposed. Finally, by means of the hermitian parity we may reduce the problem to the form considered in 44.32) - 44.35), from which the other moduli can readily be generated, using the principles by which they were eliminated. The classification of moduli according to the powers of $M^{1\kappa^*2\kappa} K^{12}$ can be determined by making use of the notion of units and dimensions introduced in IV C, however, care must be taken for the modularities that are not in general present.

We now state the third part of T 44).

$$44.31) M^{12\kappa} K^{12} = M^{1\kappa^*1\lambda} \epsilon_{\kappa}^1 = M^{1\lambda 1\lambda} \epsilon_{\lambda}^1 = 1 \quad (K^{12} \neq O^{12})$$

$$44.32) \quad M^{1\kappa^* 2\kappa} K^{12} = M^{21} \lambda^{21} = M^{2\lambda^* 1\kappa^*} \lambda^{21} = M^{11\kappa^*} \epsilon_{\lambda^*}^1 = M^{11\lambda} \epsilon_{\lambda}^1$$

$$44.33) \quad (M^{1\kappa^* 2\kappa} K^{12})^2 = M^{1\kappa^* 1\kappa^*} \epsilon_{\lambda^*}^1 = M^{21\kappa^*} \lambda^{21} = M^{1\lambda 1\kappa^*} \epsilon_{\lambda}^1 = M^{11} \epsilon_{\lambda}^1$$

$$44.34) \quad (M^{1\kappa^* 2\kappa} K^{12})^3 = M^{2\kappa 1\kappa^*} \lambda^{21} = M^{11\kappa^*} \epsilon_{\lambda}^1$$

$$44.35) \quad (M^{1\kappa^* 2\kappa} K^{12})^4 = M^{1\kappa^* 1\kappa^*} \epsilon_{\lambda}^1.$$

The proof of T 44) is as follows:

44.11) is proved in 21) and 42.2).

44.12): Since $\lambda^{21} M^{21}$, we have, by 21.5), the following modularities:

$$M^{21} \cdot M^{21\lambda} \cdot M^{2\lambda^* 1} \cdot M^{2\lambda 1\lambda} \cdot M^{2\lambda^* 1\lambda^*} \cdot M^{21\lambda} \cdot M^{2\lambda 1} \cdot M^{2\lambda 1\lambda^*}.$$

From 42.2), $\lambda^{21} = J^{1\kappa^*} J^{2\kappa} K^{*21} K^{12} K^{*21}$. Therefore, by 21.2), 41.13), and III A, T II, we secure

$$M^{2\kappa 1} \cdot M^{21\kappa^*} \cdot M^{2\kappa 1\kappa} \cdot M^{2\kappa^* 1\kappa^*} \cdot M^{2\kappa 1\kappa^*}.$$

Finally, $\lambda^{21} M^{2\lambda 1\lambda}$ and $\epsilon_{\lambda}^2 = \epsilon_{\kappa^*}^2 M^{2\kappa 2\kappa}$; hence by 13), $\lambda^{21} M^{2\kappa 1\lambda}$.

By hermitian parity, we have $\lambda^{21} M^{2\lambda^* 1\lambda^*}$. That, in general, λ^{21} is not modular relative to $(\epsilon_{\lambda^*}^2 \epsilon_{\lambda}^1)$ is shown by the following example: $\mathcal{U} = \mathcal{R} \cdot \mathcal{P}' = \mathcal{P}^2 = \mathcal{P}^{\text{III}} \cdot \epsilon^1 = \epsilon^2 = \delta \cdot \Theta \equiv (n|n \neq 0) \cdot K^{12} \equiv \delta_{\Theta}$.

44.13) follows from 44.11), 21.6), and 26.5).

Since $\epsilon_{\lambda}^1 = J^2 \lambda^{*12} \lambda^{21}$, the results in 44.14) follows immediately from 44.12). This completes the proof of the first part of T 44).

In 44.2), the equivalence between (1) and (4) follows from 42.12) and 35.62); and the equivalence between (3) and (4) follows from 21.6). The equalities in (6) follows from (4), 42.12), and 32.4). Next, we shall show that

$$K^{12} M^{12} \cdot K^{12} M^{1\lambda 2\lambda^*} \cdot K^{12} M^{1\lambda 2\kappa^*} \cdot K^{12} M^{1\lambda 2\kappa} \cdot \epsilon_{\kappa}^{1\lambda 1\lambda} \cdot K^{12} M^{12}$$

$M^{2\lambda^* 1\kappa^* \lambda^{21}} \leq M^{2\lambda^* 1\lambda^{21}} \cdot M^{11\kappa^* \epsilon_{\lambda^*}^1} = M^{11\kappa^* \epsilon_{\lambda^*}^1}$. (4) In view of the equation $\epsilon_{\lambda^*}^1 = J^{1\lambda} \epsilon_{\lambda}^1 \epsilon_{\lambda^*}^1$, we secure $M^{11\kappa^* \epsilon_{\lambda^*}^1} \leq M^{11\lambda} \epsilon_{\lambda}^1 \cdot M^{1\lambda 1\kappa^* \lambda^{21}} = M^{11\lambda} \epsilon_{\lambda}^1$. (5) Finally, by definition $\epsilon_{\lambda}^1 = J^2 \lambda^{*12} \lambda^{21}$, and hence we have $M^{11\lambda} \epsilon_{\lambda}^1 \leq M^{1\lambda 2} \lambda^{*12} M^{21\lambda^{21}} = M^{21\lambda^{21}}$.

To prove the equalities in 44.33), we find from 44.32) the fact that $M^{21\lambda^{21}} = M^{1\kappa^* 2\kappa K^{12}}$. Hence it suffices to show that

$$M^{1\kappa^* 1\kappa^* \epsilon_{\lambda^*}^1} \leq M^{21\kappa^* \lambda^{21}} \leq M^{1\lambda 1\kappa^* \epsilon_{\lambda}^1} \leq (M^{21\lambda^{21}})^2 \leq M^{11} \epsilon_{\lambda}^1 \leq M^{1\kappa^* 1\kappa^* \epsilon_{\lambda^*}^1}.$$

(1) From $\epsilon_{\lambda^*}^1 = \epsilon_{\kappa}^1 = J^2 K^{12} \lambda^{21}$, we have $M^{1\kappa^* 1\kappa^* \epsilon_{\lambda^*}^1} \leq M^{1\kappa^* 2\kappa K^{12}} M^{21\kappa^* \lambda^{21}} = M^{21\kappa^* \lambda^{21}}$. (2) Since $\lambda^{21} = J^{1\lambda} \lambda^{21} \epsilon_{\lambda}^1$, it follows that $M^{21\kappa^* \lambda^{21}} \leq M^{21\lambda} \lambda^{21} M^{1\lambda 1\kappa^* \epsilon_{\lambda}^1} = M^{1\lambda 1\kappa^* \epsilon_{\lambda}^1}$. (3) From $\epsilon_{\lambda}^1 = J^1 \epsilon_{\lambda}^1 \epsilon_{\lambda^*}^1 = J^1 J^2 \lambda^{*12} \lambda^{21} \epsilon_{\lambda^*}^1$ we have $M^{1\lambda 1\kappa^* \epsilon_{\lambda}^1} \leq M^{1\lambda 2} \lambda^{*12} M^{21\lambda^{21}} M^{11\kappa^* \epsilon_{\lambda^*}^1} = (M^{21\lambda^{21}})^2$. (4) Now $\lambda^{21} M^{21\lambda} \cdot \epsilon_{\lambda}^1 M^{11} \cdot \lambda^{21} M^{21}$. Thus by III A, T I, we have the inequality, $M^{21\lambda^{21}} \leq M^{21\lambda} \lambda^{21} / \sqrt{M^{11} \epsilon_{\lambda}^1}$. But $M^{21\lambda} \lambda^{21} = 1$, and hence $N^{21\lambda^{21}} \leq M^{11} \epsilon_{\lambda}^1$. (5) Finally, by the equation $\epsilon_{\lambda}^1 = J^{1\kappa^*} \epsilon_{\kappa}^1 \epsilon_{\kappa}^1$, we secure by 21.8), $M^{11} \epsilon_{\lambda}^1 \leq M^{11\kappa^*} \epsilon_{\kappa}^1 \cdot M^{1\kappa^* 1} \epsilon_{\kappa}^1 = M^{1\kappa^* 1\kappa^*} \epsilon_{\kappa}^1 = M^{1\kappa^* 1\kappa^* \epsilon_{\lambda^*}^1}$.

We prove 44.35) before 44.34) so as to be able to use the result in the proof of the latter. For 44.35), we find from $\epsilon_{\lambda}^1 = J^{1\lambda} \epsilon_{\lambda}^1 \epsilon_{\lambda}^1$ that $M^{1\kappa^* 1\kappa^* \epsilon_{\lambda}^1} \leq M^{1\kappa^* 1\lambda} \epsilon_{\lambda}^1 \cdot M^{1\lambda 1\kappa^* \epsilon_{\lambda}^1} = (M^{1\kappa^* 2\kappa K^{12}})^4$ by virtue of 44.33). On the other hand, we find from III A, T I, and $\epsilon_{\lambda}^1 M^{1\kappa^* 1\kappa^*}$ that $M^{21\kappa^* \lambda^{21}} \leq M^{21\lambda} \lambda^{21} / \sqrt{M^{1\kappa^* 1\kappa^* \epsilon_{\lambda}^1}}$. But by 44.33), $M^{21\kappa^* \lambda^{21}} = N^{1\kappa^* 2\kappa K^{12}}$, and by 21.2), $M^{21\lambda} \lambda^{21} = 1$; hence we have $(M^{1\kappa^* 2\kappa K^{12}})^4 \leq M^{1\kappa^* 1\kappa^* \epsilon_{\lambda}^1}$.

For 44.34), we shall show

$$(M^{1\kappa^* 2\kappa K^{12}})^3 \leq M^{11\kappa^*} \epsilon_{\lambda}^1 \leq M^{2\kappa 1\kappa^* \lambda^{21}} \leq (M^{1\kappa^* 2\kappa K^{12}})^3.$$

(1) From $\epsilon_{\lambda}^1 = J^1 \epsilon_{\lambda}^1 \epsilon_{\lambda}^1$, we secure from 44.35) that $(M^{1\kappa^* 2\kappa K^{12}})^4 = M^{1\kappa^* 1\kappa^*} \epsilon_{\lambda}^1 \leq M^{1\kappa^* 1} \epsilon_{\lambda}^1 \cdot M^{11\kappa^*} \epsilon_{\lambda}^1$. Hence by 44.32), it follows that $(M^{1\kappa^* 2\kappa K^{12}})^3 \leq M^{11\kappa^*} \epsilon_{\lambda}^1$. (2) From the relation $\epsilon_{\lambda}^1 = J^{1\kappa^*} \epsilon_{\kappa}^1 \epsilon_{\kappa}^1 =$

$= J^{1\kappa*} J^{2\kappa} K^{12} K^{*21} \epsilon_{\kappa}^1 = J^{2\kappa} K^{12} \lambda^{21}$, we obtain $M^{11\kappa*} \epsilon_{\lambda}^1 \leq M^{12\kappa} K^{12} M^{2\kappa' \kappa*} \lambda^{21} = M^{2\kappa' 1\kappa*} \lambda^{21}$. (3) From the formula for λ^{21} given in 42.2), we secure $M^{2\kappa' 1\kappa*} \lambda^{21} \leq (M^{1\kappa*} 2\kappa K^{12})^3$.

This completes the demonstration of T 44).

§ 14

Theorem 45) is devoted to the study of necessary and sufficient conditions for λ^{21} to be a reciprocal of K^{12} .

T 45.1) $\lambda^{21} = \text{Rec } K^{12}$

$$\sim. 45.11) \lambda^{21} M^{2\kappa* 1\kappa} \cdot J^2 K^{12} \lambda^{21} = \epsilon_{\kappa}^1 \cdot J^1 \lambda^{21} K^{12} = \epsilon_{\kappa*}^2$$

$$\sim. 45.12) \lambda^{21} M^{2\kappa* 1\kappa} \cdot J^1 J^2 K^{12} \lambda^{21} K^{12} = K^{12}.$$

$$\sim. 45.13) \lambda^{21} M^{2\kappa* 1} \cdot J^2 K^{12} = \epsilon_{\kappa}^1.$$

$$\sim. 45.14) \lambda^{21} T^{2\kappa* 1\kappa} \cdot J^1 \lambda^{21} K^{12} = \epsilon_{\kappa*}^2 \cdot J^2 K^{12} \lambda^{21} = \epsilon_{\kappa}^1$$

$$\cdot J^1 \lambda^{21} \mathcal{M}' < \mathcal{M}^{2\kappa} \cdot J^2 \lambda^{*12} \mathcal{M}^2 < \mathcal{M}^{1\kappa*}$$

$$\sim. 45.15) \lambda^{21} \text{ cols } M^{2\kappa*} \cdot J^2 \lambda^{*12} \mathcal{M}^2 < \mathcal{M}^{1\kappa*} \cdot J^2 K^{12} \lambda^{21} = \epsilon_{\kappa}^1.$$

T 45.2) $\mathcal{M}'_{\cap(1\kappa' \kappa*)} < \mathcal{M}'_0 < \mathcal{M}^{1\kappa} \cdot \mathcal{M}^2_{\cap(2\kappa* 2\kappa)} < \mathcal{M}^2_0 < \mathcal{M}^{2\kappa*} ;) : \lambda^{21} = \text{Rec } K^{12}$

$$:\sim: 45.21) \lambda^{21} \overline{\text{rows } M^1} \cdot J^1 \lambda^{21} \mathcal{M}' < \mathcal{M}^2_0 \cdot J^2 K^{12} \lambda^{21} = \epsilon_{\kappa}^1$$

$$:\sim: 45.22) \lambda^{21} \text{ cols } \mathcal{M}^2_0 \cdot M^{21} \cdot J^2 K^{12} \lambda^{21} = \epsilon_{\kappa}^1$$

$$:\sim: 45.23) \lambda^{21} T^{21} \cdot J^2 \lambda^{*12} \mathcal{M}^2 < \mathcal{M}'_0 \cdot J^1 \lambda^{21} \mathcal{M}' < \mathcal{M}^2_0$$

$$\cdot J^2 J^1 K^{12} \lambda^{21} K^{12} = K^{12}$$

$$:\sim: 45.24) \lambda^{21} \text{ cols } \mathcal{M}^2_0 \cdot J^2 \lambda^{*12} \mathcal{M}^2 < \mathcal{M}'_0 \cdot J^1 J^2 K^{12} \lambda^{21} K^{12} = K^{12}$$

$$\leadsto: 45.25) \lambda^{s1} \text{ cols } \mathcal{M}_0^2 : J^2 \lambda^{s1s} \mathcal{M}^2 \subset \mathcal{M}_0^1$$

$$: J^2 K^{12} \mu^{s_{K^*} \times 1_K} = \mu_0^1 \cdot J^1 \lambda^{s1} \mu_0^1 = \mu^{s_{K^*} \times 1_K}$$

$$\leadsto: 45.26) \lambda^{s1} M^{s_{K^*} \times 1_K} : J^2 K^{12} \mu^{s_{K^*} \times 1_K} = \mu_0^1 \cdot J^1 \lambda^{s1} \mu_0^1 = \mu^{s_{K^*} \times 1_K}$$

$$\leadsto: 45.27) \lambda^{s1} \overline{\text{rows } \mathcal{M}_0^1} : \mathcal{M}^{2\lambda^s} \subset \mathcal{M}^{2_{K^*} \times 1_K}$$

$$: J^1 \lambda^{s1} \mu_0^1 = \mu^{s_{K^*} \times 1_K} \cdot J^2 K^{12} \mu^{s_{K^*} \times 1_K} = \mu_0^1$$

When $K^{12} M^{12}$, then $\mathcal{M}_{\Omega(1_{K^*})}^1 = \mathcal{M}^{1_{K^*}}$, and $\mathcal{M}_{\Omega(2_{K^*} \times 1_K)}^2 = \mathcal{M}^{2_{K^*}}$, and it follows at once that every one of the preceding statements is equivalent to E. H. Moore's definition of a modular reciprocal for a modular matrix given in IV C.

The scheme of demonstration will be as follows:

$$0) \rightarrow 45.11) \rightarrow 45.12) \rightarrow 45.13) \rightarrow 0),$$

$$0) \rightarrow 45.14) \rightarrow 45.15) \rightarrow 0),$$

$$0) \rightarrow 45.21) \rightarrow 45.22) \rightarrow 0),$$

$$0) \rightarrow 45.23) \rightarrow 45.24) \rightarrow 45.12),$$

$$0) \rightarrow 45.25) \rightarrow 45.26) \rightarrow 45.12),$$

$$0) \rightarrow 45.27) \rightarrow 45.12).$$

In fact, it is evident that $0) \rightarrow 45.11) \rightarrow 45.12)$. For $45.12) \rightarrow 45.13)$, we note from IV A theory that $K^{12} = J^1 \epsilon_K^1 K^{12}$, and hence $J^1(\epsilon_K^1 - J^2 K^{12} \lambda^{s1}) K^{12} = 0^{12}$. Now $J^2 K^{12} \lambda^{s1} \overline{\text{rows } M^{1_K}}$, and hence $(\epsilon_K^1 - J^2 K^{12} \lambda^{s1}) \overline{\text{rows } M^{1_K}}$. Since $K^{12} \text{ comp cols } \mathcal{M}^{1_K}$, it follows that $J^2 K^{12} \lambda^{s1} = \epsilon_K^1$.

$45.13) \rightarrow 0)$: Since $K^{12} M^{1_{K^*} \times s}$, it follows from the second condition in 45.13) that $\epsilon_K^1 M^{1_{K^*} \times 1}$ and hence $K^{12} M^{1_{K^*} \times s_K}$. By 42.2) there exists $\lambda_1^{s1} = \text{Rec } K^{12}$. Thus $\epsilon_K^1 = J^2 K^{12} \lambda^{s1} = J^2 K^{12} \lambda_1^{s1}$, which gives $J^2 K^{12}(\lambda^{s1} - \lambda_1^{s1}) = 0^{11}$. Since $(\lambda^{s1} - \lambda_1^{s1}) \text{ cols } M^{s_{K^*} \times}$ and $K^{12} \text{ comp rows } \mathcal{M}^{2_{K^*}}$, it follows that $\lambda^{s1} = \lambda_1^{s1}$, which proves that 45.13) implies 0).

That $0) \rightarrow 45.14)$ follows from 42.13) and its hermitian parity. That $45.14) \rightarrow 45.15)$ is evident.

$45.15) \rightarrow 0)$: From the first two conditions in 45.15) and the generalized Hellinger-Toeplitz theorem, we find that λ^{s1} is $M^{2 \times 1 \times 1}$, which combined with the relation $J^2 K^{12} \lambda^{s1} = \varepsilon_\kappa^1$, gives $\varepsilon_\kappa^1 M^{1 \times 1 \times 1}$, and hence $K^{12} M^{1 \times 1 \times 2 \times \kappa}$ by 21.6). Therefore by 42.2), there exists $\lambda_s^{s1} = \text{Rec } K^{12}$. It may be shown that $\lambda_s^{s1} = \lambda^{s1}$ by reasoning as in the proof of $45.13) \rightarrow 0)$.

That $0) \rightarrow 45.21)$ follows immediately from the hermitian parity of 42.13), for

$$J^1 \lambda^{s1} \mathcal{M}' \subset \mathcal{M}^{2 \times \kappa} = \mathcal{M}_{\cap (2 \times \kappa \times 2 \times \kappa)}^2 \subset \mathcal{M}_\kappa^2.$$

$45.21) \rightarrow 45.22)$: From the first and second conditions in 45.21), we can show, by virtue of the generalized Hellinger-Toeplitz theorem, that λ^{s1} is M^{s1} .

$45.22) \rightarrow 0)$: Since $K^{12} M^{1 \times 1 \times 2}$ and $\lambda^{s1} M^{s1}$, it follows from the last condition in 45.22) that $\varepsilon_\kappa^1 M^{1 \times 1 \times 1}$, which implies K^{12} is $M^{1 \times 1 \times 2 \times \kappa}$ by 21.6). Hence by 42.2), there exists $\lambda_s^{s1} = \text{Rec } K^{12}$. It may be shown that $\lambda_s^{s1} = \lambda^{s1}$ as in the proof of $45.13) \rightarrow 0)$.

To prove $0) \rightarrow 45.23)$, we use the same reasoning as in the proof of $0) \rightarrow 45.21)$ and secure the second and third conditions. Note that $\lambda^{s1} M^{s1}$ follows from the first two conditions in 45.23) and the generalized Hellinger-Toeplitz theorem. It follows that $(J^1 J^2) K^{12} \lambda^{s1} K^{12}$ surely exists. Since K^{12} cols $M^{1 \times \kappa}$ and, by hypothesis, $J^2 K^{12} \lambda^{s1} = \varepsilon_\kappa^1$, the last condition follows.

That $45.23)$ implies $45.24)$ is evident.

$45.24) \rightarrow 45.12)$: From the first two conditions, we have λ^{s1} cols $M^{2 \times \kappa}$. $J^2 \lambda^{s1} \mathcal{M}^{2 \times \kappa} \subset \mathcal{M}^{1 \times \kappa}$. Hence λ^{s1} is $M^{2 \times 1 \times \kappa}$.

$0) \rightarrow 45.25)$: The first two conditions are obvious. For

the last condition, we find from the hermitian parity of 41.12) that λ^{s1} rows $\mathcal{M}^{1\kappa \times 2\kappa*}$ and from 41.13), $K^{1s} I^{1\kappa 2\kappa*}$; thus

$$\begin{aligned} J^s K^{1s} \mu^{2\kappa* \times 1\kappa} &= \mu_0^1 \cdot J^1 \lambda^{s1} \mu_0^1 = J^1 \lambda^{s1} (J^s K^{1s} \mu^{2\kappa* \times 1\kappa}) \\ &= J^s (J^1 \lambda^{s1} K^{1s}) \mu^{2\kappa* \times 1\kappa} = \mu^{2\kappa* \times 1\kappa}. \end{aligned}$$

That 45.25) implies 45.26) follows from the generalized Hellinger-Toeplitz theorem.

To prove 45.26) $\rightarrow$ 45.12), we note from 22.51) that $\mathcal{M}_0^1 \subset J^s K^{1s} \mathcal{M}^{2\kappa* \times 1\kappa}$. Thus by hypothesis,

$$\mu^{2\kappa* \times 1\kappa} \cdot \mu^{2\kappa* \times 1\kappa} = J^1 \lambda^{s1} (J^s K^{1s} \mu^{2\kappa* \times 1\kappa}),$$

and hence

$$J^s K^{1s} \mu^{2\kappa* \times 1\kappa} = J^s K^{1s} (J^1 \lambda^{s1} (J^s K^{1s} \mu^{2\kappa* \times 1\kappa})) (\mu^{2\kappa* \times 1\kappa}).$$

Since $e_{\kappa*}^s$ cols $\mathcal{M}^{2\kappa* \times 1\kappa}$ and $\lambda^{s1} M^{s1}$, it follows that

$$\begin{aligned} K^{1s} &= J^s K^{1s} e_{\kappa*}^s = J^s K^{1s} (J^1 \lambda^{s1} J^s K^{1s} e_{\kappa*}^s) \\ &= J^s K^{1s} J^1 \lambda^{s1} K^{1s} = J^1 J^s K^{1s} \lambda^{s1} K^{1s}. \end{aligned}$$

9) $\rightarrow$ 45. 27) follows by 42.14)_H and 41.12) that $\lambda^{s1} M^{s1}$.

45.27) $\rightarrow$ 45.12): Applying the generalized Hellinger-Toeplitz theorem to the first two conditions in 45.27), we obtain $\lambda^{s1} M^{2\kappa* \times 1\kappa}$ and hence $\lambda^{s1} M^{s1}$. Since K^{1s} cols $\mathcal{M}_0^1$ and $J^1 \lambda^{s1} K^{1s}$ is by cols $\mathcal{M}^{2\kappa* \times 1\kappa}$, we have from the third condition of 45.27) that $K^{1s} = J^s K^{1s} (J^1 \lambda^{s1} K^{1s}) = J^1 J^s K^{1s} \lambda^{s1} K^{1s}$.

§ 15

Theorem 46) studies the spaces associated with K^{1s} which is modular relative to $(e_{\kappa*}^1 e_{\kappa}^2)$.

T 46) $\lambda^{21} = \text{Rec } K^{12} \quad ;):$

$$46.1) \quad \mathcal{M}^{2\lambda} = \mathcal{M}^{2\kappa*} \left(\begin{smallmatrix} 2 & \lambda^* \\ 1 & \kappa^* \end{smallmatrix} \right) \mathcal{M}^{1\kappa \kappa^* 2\kappa*} = \mathcal{M}^{1\lambda}$$

$$46.2) \quad \mathcal{M}^{1\lambda*} = \mathcal{M}^{1\kappa} \left(\begin{smallmatrix} 1 & \lambda \\ 2 & \kappa \end{smallmatrix} \right) \mathcal{M}^{2\kappa* \kappa^* 1\kappa} = \mathcal{M}^{2\lambda*}$$

$$46.3) \quad (1) \quad \mathcal{M}^{2\kappa*} \left(\begin{smallmatrix} 2 & \kappa \\ 1 & \lambda \end{smallmatrix} \right) \mathcal{M}^{1\lambda} \sim (2) \quad \mathcal{M}^{1\lambda} = \mathcal{M}^{1\kappa*}$$

$$\sim (3) \quad \mathcal{M}^{1\kappa} = \mathcal{M}^{1\lambda} \quad \sim (4) \quad \lambda^{21} M^{2\lambda* 1\lambda}$$

$$46.4) \quad \mu^{1\lambda} \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mu^{2\kappa*} \cdot \mu^{1\kappa} \left(\begin{smallmatrix} 1 & \lambda \\ 2 & \kappa \end{smallmatrix} \right) \mu^{2\lambda*} \cdot J^1 \overline{\mu^{1\lambda}} \mu^{1\kappa} = J^2 \overline{\mu^{2\kappa*}} \mu^{2\lambda*}$$

$$46.51) \quad \mathcal{M}_0^{2\kappa*} \left(\begin{smallmatrix} 2 & \lambda^* \\ 1 & \kappa^* \end{smallmatrix} \right) \mathcal{M}_0^{1\lambda} \cdot \mathcal{M}_0^{1\lambda L J^1} \cdot \mathcal{M}_0^{2\kappa* L J^2}$$

$$46.52) \quad \mathcal{M}_0^{2\kappa*} \left(\begin{smallmatrix} 2 & \lambda^* \\ 1 & \kappa^* \end{smallmatrix} \right) \mathcal{M}_0^{1\lambda} \cdot \mathcal{M}_0^{2\kappa* L J^2} \cdot \mathcal{M}_0^{1\lambda L J^1} \sim \lambda^{21} M^{2\lambda* 1\lambda}$$

$$46.6) \quad \mathcal{M}_1^{1\lambda} \left(\begin{smallmatrix} 1 & \kappa^* \\ 2 & \lambda^* \end{smallmatrix} \right) \mathcal{M}_1^{2\kappa*} \cdot \mathcal{M}_2^{1\kappa} \left(\begin{smallmatrix} 1 & \lambda \\ 2 & \kappa \end{smallmatrix} \right) \mathcal{M}_2^{2\lambda*} :$$

$$46.61) \quad \mathcal{M}_1^{1\lambda} O' \mathcal{M}_2^{1\kappa} \sim \mathcal{M}_1^{2\kappa*} O^2 \mathcal{M}_2^{2\lambda*}$$

$$46.62) \quad \mathcal{M}_2^{1\kappa} = \cap (\mathcal{M}^{1\kappa}, O' \mathcal{M}_1^{1\lambda}) \sim \mathcal{M}_2^{2\lambda*} = \cap (\mathcal{M}^{2\lambda*}, O^2 \mathcal{M}_1^{2\kappa*})$$

$$: \mathcal{M}_1^{2\kappa*} = \cap (\mathcal{M}^{2\kappa*}, O^2 \mathcal{M}_2^{2\lambda*}) \sim \mathcal{M}_1^{1\lambda} = \cap (\mathcal{M}^{1\lambda}, O' \mathcal{M}_2^{1\kappa})$$

From 41.13), 42.14), 42.12), 35.11), and 35.12), we secure 46.1) and 46.2).

In 46.3), we note the first equivalence follows from 42.13), 42.12), and 35.21). To complete the proof, we shall show $(4) \rightarrow (2) \rightarrow (3) \rightarrow (4)$. Now if (4) holds, then by 44.2), (5), $\varepsilon_{\kappa^*}^{1\lambda}$ is $M^{1\lambda 1\lambda}$, and hence $\mathcal{M}^{1\kappa*} \subset \mathcal{M}^{1\lambda}$. Thus by 42.13), we have (2). If (2) holds, then, since $\mathcal{M}^{1\kappa} = \mathcal{M}^{1\lambda*}$, we secure $\mathcal{M}^{1\kappa} \supset \mathcal{M}^{1\lambda} = \mathcal{M}^{1\kappa*}$. Hence by 42.13), it follows that $\mathcal{M}^{1\kappa} = \mathcal{M}^{1\lambda}$. Finally, from (3), we have $\mathcal{M}^{1\lambda} = \mathcal{M}^{1\kappa} = \mathcal{M}^{1\lambda*}$; thus by 21.6), we secure (4).

46.4) follows from 42.12), 42.13), and 35.5).

¹ Since (4) is its own hermitian parity, (1), (2), and (3) are equivalent to their hermitian parities.

To prove 46.51), we note that the linearity is obvious. For the J^2 -closure property, let us consider $\mu^{2\kappa^*} = \bigcup_{n=1}^\infty \mu_{on}^{2\kappa^*}$. By hypothesis,

$$\exists \{ \mu_{on}^{1\lambda} \} \ni J^2 \lambda^{*12} \mu_{on}^{2\kappa^*} = \mu_{on}^{1\lambda}.$$

Then

$$\begin{aligned} \overline{\bigcup_n M^1 \mu_{on}^{1\lambda}} &= \overline{\bigcup_n M^1 J^2 \lambda^{*12} \mu_{on}^{2\kappa^*}} = \overline{\bigcup_n M^{12} \lambda^{*12} M^2 \mu_{on}^{2\kappa^*}} \\ &\leq M^{21} \lambda^{21} \overline{\bigcup_n M^2 \mu_{on}^{2\kappa^*}} < \infty \end{aligned}$$

and

$$\bigcup_{n=1}^\infty \mu_{on}^{1\lambda} = \bigcup_{n=1}^\infty J^2 \lambda^{*12} \mu_{on}^{2\kappa^*} = J^2 \lambda^{*12} \left(\bigcup_{n=1}^\infty \mu_{on}^{2\kappa^*} \right) = J^2 \lambda^{*12} \mu^{2\kappa^*}.$$

Consequently, $\mu_o^{1\lambda} \equiv \bigcup_{n=1}^\infty \mu_{on}^{1\lambda}$ is in $\mathcal{M}_o^{1\lambda}$, since the latter is LJ^1 ; furthermore,

$$\begin{aligned} J^1 K^{*21} \mu_o^{1\lambda} &= J^1 K^{*21} (J^2 \lambda^{*12} \mu^{2\kappa^*}) = J^2 (J^1 K^{*21} \lambda^{*12}) \mu^{2\kappa^*} \\ &= J^2 e_{\kappa^*}^2 \mu^{2\kappa^*} = \mu^{2\kappa^*}. \end{aligned}$$

Thus $\mu^{2\kappa^*}$ is in $\mathcal{M}_o^{2\kappa^*}$.

For 46.52), we first prove the condition is necessary. By 46.1) and hypothesis, it is evident that $\mathcal{M}^{1\lambda} \supset LJ^1$. Now $\lambda^{21} M^{21}$; hence by Cor. to T 21), $\lambda^{21} M^{2\lambda^* 1\lambda}$. Conversely, if $\lambda^{21} M^{2\lambda^* 1\lambda}$, then $K^{12} M^{12}$ and $K = \text{Rec } \lambda^{21}$ by 44.2). The mutual parity of 46.51 gives

$$\mathcal{M}_o^{2\kappa} \left(\begin{smallmatrix} 2 & \lambda^* \\ 1 & \kappa^* \end{smallmatrix} \right) \mathcal{M}_o^{1\lambda^*} \cdot \mathcal{M}_o^{2\kappa} \supset LJ^2 \cdot \mathcal{M}_o^{1\lambda^*} \supset LJ^1$$

But $\mathcal{M}^{2\kappa} = \mathcal{M}^{2\kappa^*} \cdot \mathcal{M}^{1\lambda^*} = \mathcal{M}^{1\lambda}$ by 21.5) and 21.6). Thus the condition is also sufficient.

By 42.12) and 42.13), 46.6) is an instance of 35.7).

§ 16

When λ^{21} is a reciprocal of K^{12} , and λ^{22} that of K^{22} , ex-

amples show that the composite matrix $\lambda^{s1} \equiv J^2 \lambda^{s2} \lambda^{21}$ is not, in general, a reciprocal of the composite matrix $K^{1s} \equiv J^2 K^{12} K^{2s}$. In theorem 47), we state a sufficient condition for λ^{s1} to be a reciprocal of K^{1s} , which is assumed to be of T^{1s} .

$$T \ 47) \ \left. \begin{aligned} \beta^s \cdot \epsilon^s \cdot K^{2s} T^{2s} \cdot \lambda^{s2} &= \text{Rec } K^{2s} \cdot \lambda^{s1} = \text{Rec } K^{12} \\ K^{1s} \equiv J^2 K^{12} K^{2s} T^{1s} \cdot \lambda^{s1} &\equiv J^2 \lambda^{s2} \lambda^{21} \\ \mathcal{M}^{2\kappa^{23}} = \mathcal{M}^{2\kappa^{21}} \cdot \mathcal{M}^{1\kappa^{12}} &= \mathcal{M}^{1\kappa^{13}} \cdot \mathcal{M}^{3\kappa^{32}} = \mathcal{M}^{3\kappa^{31}} \end{aligned} \right) \cdot \lambda^{s1} = \text{Rec } K^{1s}$$

In case $K^{1s} M^{1s}$, then the condition $\mathcal{M}^{2\kappa^{23}} = \mathcal{M}^{2\kappa^{21}}$ implies the other two space equalities in the hypothesis.

On account of the LJ^2 property of $\mathcal{M}^{2\kappa^{23}}$ and $\mathcal{M}^{2\kappa^{21}}$, we may replace the space equality $\mathcal{M}^{2\kappa^{23}} = \mathcal{M}^{2\kappa^{21}}$ by $\epsilon_{\kappa^{23}}^2 = \epsilon_{\kappa^{21}}^2$. Similarly for the other two space equalities.

To prove the theorem, let us use T 45.13) on p. 78. Now $\lambda^{s2} = \text{Rec } K^{2s}$, then $\lambda^{s2} M^{2\kappa^{22}}^2$. Also $\lambda^{21} M^{21}$. Hence by the composition of modularity, we have $\lambda^{s1} = J^2 \lambda^{s2} \lambda^{21} M^{3\kappa^{32}}^1$. Consequently, by hypothesis, we have $\lambda^{s1} M^{3\kappa^{31}}^1$. Furthermore,

$$\begin{aligned} J^2 K^{1s} \lambda^{s1} &= J^2 K^{1s} (J^2 \lambda^{s2} \lambda^{21}) = J^2 (J^2 K^{1s} \lambda^{s2}) \lambda^{21} \\ &= J^2 (J^2 (J^2 K^{12} K^{2s}) \lambda^{s2}) \lambda^{21}. \end{aligned}$$

Now by IV A theory, since $K^{1s} T^{1s}$, we have $K^{1s} T^{1\kappa^{12}} s_{\kappa^{31}}$. Hence by hypothesis, K^{1s} is of $T^{1\kappa^{12}} s_{\kappa^{32}}$. Since K^{2s} has the property of $I^{2\kappa^{23}} 3_{\kappa^{32}}$ and $[\text{rows } K^{12}] \subset \mathcal{M}^{2\kappa^{23}} \kappa^{*22} 3_{\kappa^{32}}$, we have

$$J^2 (J^2 K^{1s} K^{2s}) \lambda^{s2} = J^2 K^{1s} (J^2 K^{2s} \lambda^{s2}) = J^2 K^{1s} \epsilon_{\kappa^{23}}^2 = J^2 K^{1s} \epsilon_{\kappa^{*21}}^2 = K^{1s}.$$

Therefore,

$$J^2 K^{1s} \lambda^{s1} = J^2 K^{1s} \lambda^{21} = \epsilon_{\kappa^{12}}^1 = \epsilon_{\kappa^{13}}^1.$$

V. NON-MODULAR RECIPROCAL OF MODULAR MATRICES

In the present chapter, we shall give a less restricted definition of a reciprocal λ^{21} for a given modular matrix K^{12} . In fact, this may be considered as an inverse problem of the preceding chapter. On account of the weaker conditions imposed upon λ^{21} , the mutuality property is not preserved. The definitions given here are proved to be equivalent to those given in Chapter III. Furthermore, the matrix K^{12} in our basis is the reciprocal of λ^{21} in the sense defined in Chapter IV.

The basis for the present study is

$$u^D \cdot p' \cdot p^z \cdot \varepsilon'^{PH} \cdot \varepsilon^z{}^{PH} \cdot K'^{12} M^{12}$$

§ 17

$$D \ 51.1r \quad \lambda^{21} = r \text{ rec } K^{12} \quad \equiv \quad \lambda^{21} \text{ cols } M^{2\kappa*} \cdot \text{comp cols } \mathcal{M}^{2\kappa*}$$

$$\cdot J^2 K^{12} \lambda^{21} = \varepsilon_{\kappa}^1$$

$$D \ 51.1\lambda \quad \lambda^{21} = \lambda \text{ rec } K^{12} \quad \equiv \quad \lambda^{21} \overline{\text{rows}} M^{1\kappa} \cdot \text{comp } \overline{\text{rows}} \mathcal{M}^{1\kappa}$$

$$\cdot J^1 \lambda^{21} K^{12} = \varepsilon_{\kappa}^2.$$

$$D \ 51.2) \quad \lambda^{21} = \text{rec } K^{12} \quad \equiv \quad \lambda^{21} r \text{ rec } K^{12} \cdot \lambda \text{ rec } K^{12}$$

$$D \ 52.1r \quad \lambda^{21} = r \text{ rec } K^{12} \quad \equiv \quad \lambda^{21} \text{ cols } M^B \cdot J^2 K^{12} \lambda^{21} = \varepsilon^1$$

$$\cdot J^1 \lambda^{21} \mathcal{M}^{1\kappa} = \mathcal{M}^{2\kappa}$$

$$D \ 52.1\lambda \quad \lambda^{21} = \lambda \text{ rec } K^{12} \quad \equiv \quad \lambda^{21} \overline{\text{rows}} M^1 \cdot J^1 \lambda^{21} K^{12} = \varepsilon^2$$

$$\cdot J^2 \lambda^{21} \lambda^{*12} \mathcal{M}^{2\kappa*} = \mathcal{M}^{1\kappa*}$$

$$D\ 52.2) \quad \lambda^{21} = \text{rec } K^{12} \cdot \Xi \cdot \lambda^{21} \text{ r rec } K^{12} \cdot \lambda \text{ rec } K^{12}$$

Properties of a right reciprocal are given in the following theorem:

T 51.1) $\lambda^{21} = \text{r rec } K^{12} \cdot \lambda^{21} = \text{r rec } K^{12}$ in the sense defined in Chapter III.

$$51.2) \quad \Xi \lambda^{21} = \text{r rec } K^{12} \cdot \sim \cdot e_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \text{comp cols } \mathcal{M}'^{\kappa*}$$

$$51.3) \quad \Xi \lambda^{21} = \text{r rec } K^{12} \cdot \cdot \cdot$$

$$51.31) \quad e_{\kappa}^1 I^{1\kappa 1\kappa*}$$

$$51.32) \quad \mathcal{M}'^{\kappa*} e_{\kappa}^1 = \mathcal{M}'^{\kappa*} \kappa^{*2\kappa} = J^2 K^{12} \mathcal{M}^{2\kappa}$$

$$51.33) \quad J^{1\kappa*} e_{\kappa}^1 \mathcal{M}'^{\kappa*} e_{\kappa}^1 = \mathcal{M}'^{\kappa}$$

$$51.34) \quad J^{1\kappa*} K^{*21} e_{\kappa}^1 = \text{r rec } K^{12}$$

$$51.4) \quad \lambda^{21} = \text{r rec } K^{12} \cdot \cdot \cdot \quad 51.41) \quad J^2 \lambda^{*12} \mathcal{M}^{2\kappa} = \mathcal{M}'^{\kappa}$$

$$51.42) \quad e_{\kappa}^1 M^{1\lambda 1\lambda} \cdot \mathcal{M}'^{\kappa} \subset \mathcal{M}'^{\lambda}$$

$$51.5) \quad \lambda^{21} = \text{r rec } K^{12} \cdot \sim \cdot \lambda^{21} \text{ cols } M^{2\kappa*} \cdot J^2 K^{12} \lambda^{21} = e_{\kappa}^1$$

$$\cdot J^{1\lambda} \lambda^{21} \mathcal{M}'^{\kappa} = \mathcal{M}^{2\kappa}$$

In fact, 51.1) is evident, since $\mathcal{M}'^{\kappa} = \mathcal{M}'^{\kappa} \kappa^{*2\kappa*}$. The equivalence in 51.2) and the result in 51.34) follows from 51.1), 31.12), and 29); 51.31) follows from 51.2) and 29); 51.32) from 21.5), 24.2), and 29); 51.33) from 51.2), 51.32), 28.1), and the hermitian parity of 22.1).

To prove 51.41), we note from IV A theory that $\mathcal{M}^{2\kappa} = J^1 K^{*21} \mathcal{M}'^{\kappa}$, from which we secure

$$J^2 \lambda^{*12} \mathcal{M}^{2\kappa} = J^2 \lambda^{*12} J^1 K^{*21} \mathcal{M}'^{\kappa} = J^1 (J^2 K^{12} \lambda^{21})^* \mathcal{M}'^{\kappa}$$

$$= J^1 \varepsilon_{\kappa}^1 \mathfrak{M}^{1\kappa} = \mathfrak{M}^{1\kappa} ,$$

which proves 51.41). 51.42) follows from 26.7).

To prove 51.5), we find from 26.4) that $\mathfrak{M}^{2\lambda} = \mathfrak{M}^{2\kappa*}$, from which it follows that $\mathfrak{M}^{2\kappa} \subset \mathfrak{M}^{2\lambda}$. Hence by 51.41) and 51.42),

$$\begin{aligned} J^{1\lambda} \lambda^{s1} \mathfrak{M}^{1\kappa} &= J^{1\lambda} \lambda^{s1} J^s \lambda^{*1s} \mathfrak{M}^{s\kappa} = J^s (J^{1\lambda} \lambda^{s1} \lambda^{*1s}) \mathfrak{M}^{s\kappa} \\ &= J^s \varepsilon_{\lambda}^s \mathfrak{M}^{s\kappa} = \mathfrak{M}^{s\kappa} . \end{aligned}$$

This proves the condition is necessary. For the sufficiency, we find from 26.1) that ε_{κ}^1 cols $M^{1\kappa*}$. Further, from 26.7), $\mathfrak{M}^{1\kappa} \subset \mathfrak{M}^{1\lambda}$, and hence $\mathfrak{M}^{s\kappa} \subset J^{1\lambda} \lambda^{s1} \mathfrak{M}^{1\lambda} = \mathfrak{M}^{s\lambda}$. Thus, by 21.5) and 28.31), ε_{κ}^1 comp cols $\mathfrak{M}^{1\kappa*}$. From 26.2), we obtain $\lambda^{s1} = J^{1\kappa*} K^{*s1} \varepsilon_{\kappa}^1$. Hence by 26.6) λ^{s1} comp cols $\mathfrak{M}^{s\kappa*}$.

Theorem 52) states properties of a reciprocal λ^{s1} .

$$T \ 52.1) \ \lambda^{s1} = \text{rec } K^{1s} \ .).$$

$$52.11) \ \lambda^{s1} = \text{rec } K^{1s} \text{ in the sense defined in Chapter III.}$$

$$52.12) \ K^{1s} = \text{Rec } \lambda^{s1} \text{ in the sense defined in Chapter IV.}$$

$$\begin{aligned} 52.2) \ \exists \ \lambda^{s1} = \text{rec } K^{1s} \ . \sim. \ \varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \text{comp cols } \mathfrak{M}^{1\kappa*} \\ \cdot \ \varepsilon_{\kappa^*}^s \overline{\text{rows } M^s} \cdot \text{comp } \overline{\text{rows } \mathfrak{M}^{s\kappa}} \\ .) \cdot J^{1\kappa*} K^{*s1} \varepsilon_{\kappa}^1 = J^{s\kappa} \varepsilon_{\kappa^*}^s K^{*s1} = \text{rec } K^{1s} \end{aligned}$$

$$52.3) \ \lambda^{s1} = \text{rec } K^{1s} \ .).$$

$$52.31) \ \mathfrak{M}^{1\kappa*} \subset \mathfrak{M}^{1\lambda*} = \mathfrak{M}^{1\kappa} \subset \mathfrak{M}^{1\lambda} = J^s \lambda^{*1s} \mathfrak{M}^s$$

$$52.32) \ J^{1\kappa*} K^{*s1} = J^1 \lambda^{s1} \text{ on } \mathfrak{M}^{1\kappa*}$$

$$52.33) \ (\mathfrak{M}^{1\lambda*} \lambda^{s\lambda} \varepsilon_{\lambda}^s) L J^{1\kappa*} = \mathfrak{M}^{1\kappa*}$$

$$52.34) \quad \lambda^{s1} M^{s\lambda^* 1\lambda}$$

$$52.35) \quad m_{\lambda^*}^{1\lambda^* s\lambda} = m^{1\kappa}$$

52.11) is evident from 51.1) and its hermitian parity.

To prove 52.12), we find from 52.11) and 32.23) that $e_{\kappa^*}^s = e_{\lambda}^s$ and $e_{\kappa}^1 = e_{\lambda^*}^1$. Thus by 21.5) it is evident that

$$K^{1s} M^{1\lambda^* s\lambda} \cdot J^s K^{1s} \lambda^{s1} = e_{\lambda^*}^1 \cdot J^1 \lambda^{s1} K^{1s} = e_{\lambda}^s,$$

which, by virtue of 45.11), shows that $K^{1s} = \text{Rec } \lambda^{s1}$.

52.2) follows from 51.2) and its hermitian parity for a left reciprocal.

52.31) follows from 52.12) and 42.13); 52.32) from 52.11), 32.26), and 21.5). The space-equality in 52.33) follows from 52.12) and 42.14), and the $LJ^{1\kappa^*}$ -property follows as a consequence, since $m^{1\kappa^*} LJ^{1\kappa^*}$. 52.34) follows from 52.31) and the corollary to the generalized Hellinger-Toeplitz theorem; and finally, 52.35) from 52.12) and 42.14).

The following theorem states the properties of classical reciprocals:

T 53.1) $\lambda^{s1} = r \underset{0}{\text{rec}} K^{1s} \cdot \lambda^{s1} = r \underset{0}{\text{rec}} K^{1s}$ in the sense defined in Chapter III.

$$53.2) \quad \lambda^{s1} = r \underset{0}{\text{rec}} K^{1s} \cdot$$

$$53.21) \quad \lambda^{s1} = r \underset{0}{\text{rec}} K^{1s} \text{ in the sense defined in Chapter III.}$$

$$53.22) \quad \lambda^{s1} I^{s1} \cdot T_{\text{S}}^{s1}$$

$$53.23) \quad K^{1s} = \underset{0}{\text{Rec}} \lambda^{s1} \text{ in the sense defined in Chapter IV.}$$

$$53.3) \quad \exists \lambda_r^{s1} \sim K^{1s} \text{ comp cols } m^1. e_{\kappa}^1 \text{ cols } M^{1\kappa^*} \cdot \text{comp cols } m^{1\kappa^*}$$

$$\cdot J^{1\kappa^*} K^{s1} e_{\kappa}^1 = r \underset{0}{\text{rec}} K^{1s}$$

$$53.4) \quad \mathbb{E} \lambda_{\kappa}^{21} \sim K^{12} \text{ comp cols } \mathcal{M}^1 \cdot \text{comp rows } \mathcal{M}^2$$

$$\cdot \varepsilon_{\kappa}^1 \text{ cols } M^{1\kappa*} \cdot \text{comp cols } \mathcal{M}^{1\kappa*}$$

$$\cdot \varepsilon_{\kappa*}^2 \overline{\text{rows}} M^{2\kappa} \cdot \text{comp rows } \mathcal{M}^{2\kappa}$$

$$\cdot) \cdot J^{1\kappa*} K^{221} \varepsilon_{\kappa}^1 = J^{2\kappa} \varepsilon_{\kappa*}^2 K^{221} = \text{rec } K^{12}$$

In fact, 53.1) is evident from 21.5) and 26.7). 53.21) is an immediate corollary of 53.1) and its hermitian parity. In 53.22), the I^{21} -property of λ^{21} follows from 53.21) and 33.63). The $T_{\mathbb{E}}^{21}$ - property of λ^{21} now follows from the I^{21} -property and 25.42). From 53.22) and D 42), we get 53.23). 53.3) follows from 53.1), 33.4), and 29). 53.4) is the combined result of 53.3) and its hermitian parity.

From 52.12) and 44), we secure the following theorem:

$$T \ 54) \quad \lambda^{21} = \text{rec } K^{12} \quad :):$$

$$54.11) \quad \lambda^{21} M^{21\lambda} \cdot M^{2\lambda 1\lambda} \cdot M^{2\lambda* 1} \cdot M^{2\lambda* 1\lambda*} \cdot M^{2\lambda* 1\lambda}$$

$$54.12) \quad K^{12} \text{ 15 mods, and in general } -M^{1\kappa* 2\kappa}$$

$$54.13) \quad \varepsilon_{\kappa}^1 (= \varepsilon_{\lambda*}^1) \text{ has the following 11 mods:}$$

$$M^{11}, M^{1\kappa 1\kappa}, M^{11\kappa}, M^{11\lambda}, M^{1\kappa 1\lambda}, M^{1\kappa* 1\lambda}, \\ M^{1\lambda 1\lambda}, M^{1\kappa 1}, M^{1\lambda 1}, M^{1\lambda 1\kappa}, M^{1\lambda 1\kappa*}.$$

$$54.14) \quad \varepsilon_{\kappa*}^1 \text{ 16 mods}$$

$$54.2) \quad (1) \quad \lambda^{21} M^{21}$$

$$\sim (2) \quad \lambda^{21} \text{ any one of the remaining 11 mods}$$

$$\sim (3) \quad K^{12} M^{1\kappa* 2\kappa}$$

$$\sim (4) \quad \varepsilon_{\lambda*}^1 \text{ any one of the remaining 5 mods}$$

~. (5) ε_{λ}^1 any one of the remaining 15 mods

.) . (6) $\lambda^{21} = J^{1\kappa*} J^{2\kappa} K^{*21} K^{12} K^{*21} \cdot \lambda^{21} = \text{rec } K^{12}$

$$54.31) M^{12\kappa} K^{12} = M^{1\kappa*1\lambda} \varepsilon_{\lambda*}^1 = M^{1\lambda1\lambda} \varepsilon_{\lambda}^1 = 1 \quad (K^{12} \neq 0^{12})$$

$$54.32) M^{12} K^{12} = M^{11\kappa*} \varepsilon_{\kappa*}^1 = M^{2\lambda*1\lambda} \lambda^{21} = M^{11\lambda} \varepsilon_{\lambda*}^1 = M^{1\lambda2\kappa} K^{12}$$

$$54.33) (M^{12} K^{12})^2 = M^{11} \varepsilon_{\kappa*}^1 = M^{1\lambda1\lambda} \varepsilon_{\kappa}^1 = M^{1\lambda2} K^{12} = M^{1\kappa*1\lambda} \varepsilon_{\kappa*}^1$$

$$54.34) (M^{12} K^{12})^3 = M^{1\lambda2\lambda*} K^{12} = M^{11\lambda} \varepsilon_{\kappa*}^1$$

$$54.35) (M^{12} K^{12})^4 = M^{1\lambda1\lambda} \varepsilon_{\kappa*}^1$$

§ 18

Necessary and sufficient conditions for a matrix λ^{21} to be a reciprocal of K^{12} are stated in the following theorem:

T 55) $\lambda^{21} = \text{rec } K^{12}$

:~: 55.1) $\lambda^{21} T^{2\kappa*1\kappa} \cdot \text{comp cols } \mathcal{M}^{2\kappa*} \cdot \text{comp rows } \mathcal{M}'_{\kappa}$

$$\cdot J^{2} K^{12} \lambda^{21} = \varepsilon_{\kappa}^1$$

:~: 55.2) $\lambda^{21} T^{2\kappa*1\kappa} \cdot J^{1\lambda} \lambda^{21} \mathcal{M}'_{\kappa} \supset \mathcal{M}^{2\kappa} \cdot J^{2\lambda*} \lambda^{*12} \mathcal{M}^{2\kappa*} \supset \mathcal{M}'_{\kappa*}$

$$\cdot J^{2} K^{12} \lambda^{21} = \varepsilon_{\kappa}^1$$

:~: 55.3) $\lambda^{21} T^{21} \cdot \mathcal{M}^{2\lambda} = \mathcal{M}^{2\kappa*} \cdot \mathcal{M}'_{\lambda*} = \mathcal{M}'_{\kappa} \cdot J^{2} K^{12} \lambda^{21} = \varepsilon_{\kappa}^1$

:~: 55.4) $\lambda^{21} T^{2\kappa*1\kappa} \cdot \text{comp cols } \mathcal{M}^{2\kappa*} \cdot \text{comp rows } \mathcal{M}'_{\kappa}$

$$\cdot J^1 J^2 \lambda^{21} K^{12} \lambda^{21} = \lambda^{21}$$

:~: 55.5) $\lambda^{21} T^{2\kappa*1\kappa} \cdot \text{comp cols } \mathcal{M}^{2\kappa*} \cdot \text{comp rows } \mathcal{M}'_{\kappa}$

$$: J^2 K^{12} \mu^{2\kappa*} = \mu^{1\kappa*} \cdot J^1 \lambda^{21} \mu^{1\kappa*} = \mu^{2\kappa*}$$

:~: 55.6) $\lambda^{s1} T^{s1}$. $\mathcal{M}^{s\lambda} = \mathcal{M}^{s\kappa*}$. $\mathcal{M}^{1\lambda*} = \mathcal{M}^{1\kappa}$. $J^1 \lambda^{s1} K^{1s} H$

$$: J^1 \lambda^{s1} \mu^{1\kappa*} = \mu^{s\kappa*} \cdot). \quad J^{sK^{1s}} \mu^{s\kappa*} = \mu^{1\kappa*}$$

The condition $J^1 \lambda^{s1} K^{1s} H$ in 55.6) may be replaced by $J^1 \lambda^{s1} K^{1s}$ cols $M^{s\kappa*}$.

The scheme of proof will be as follows:

$$0) \rightarrow 55.3) \rightarrow 55.1) \rightarrow 55.2) \rightarrow 0),$$

$$0) \rightarrow 55.5) \rightarrow 55.4) \rightarrow 55.1),$$

$$0) \rightarrow 55.6) \rightarrow 55.3)H.$$

That $0) \rightarrow 55.3) \rightarrow 55.1)$ follows from IV A, T III with the basis $(\mathcal{U}^D \cdot \mathcal{P}^1 \cdot \mathcal{P}^s \cdot \epsilon_{\kappa^*}^s \cdot \lambda^{s1} T^{s\kappa^{*1\kappa}})$. To prove $55.1)$ implies $55.2)$, we note that $J^1 \lambda^{s1} \mathcal{M}^{1\kappa} \supset \mathcal{M}^{s\kappa}$ follows from 51.5). From 26.1), we find ϵ_{κ}^1 cols $M^{1\kappa*}$. Thus by 26.2) and 26.6), we secure $\lambda^{s1} = J^{1\kappa*} K^{s1} \epsilon_{\kappa}^1$ and ϵ_{κ}^1 comp cols $\mathcal{M}^{1\kappa*}$. Hence by 29) and 27.27), $\epsilon_{\kappa^*}^s \overline{\text{rows}} M^{s\kappa}$. It follows from 28.34) that $J^1 \lambda^{s1} K^{1s} = \epsilon_{\kappa^*}^s$. With this relation, we secure $J^{s\lambda*} \lambda^{s1s} \mathcal{M}^{s\kappa*} \supset \mathcal{M}^{1\kappa*}$ by the hermitian parity of 51.5). To prove $55.2) \rightarrow 0)$, we find from 26.1), 26.2), and 28.31) that ϵ_{κ}^1 cols $M^{1\kappa*}$. comp cols $\mathcal{M}^{1\kappa*}$ and $\lambda^{s1} = J^{1\kappa*} K^{s1} \epsilon_{\kappa}^1$. Thus by 26.6), λ^{s1} comp cols $\mathcal{M}^{s\kappa*}$. From 29), 27.27) and 28.34), we secure $J^1 \lambda^{s1} K^{1s} = \epsilon_{\kappa^*}^s$. By similar reasoning, we secure λ^{s1} comp rows $\mathcal{M}^{1\kappa}$.

That $0) \rightarrow 55.5)$ is evident. To prove $55.5) \rightarrow 55.4)$, consider any $\mu^{s\kappa*}$. By IV A theory, $J^{sK^{1s}} \mu^{s\kappa*} M^{1\kappa*}$ and hence by hypothesis, $J^1 \lambda^{s1} (J^{sK^{1s}} \mu^{s\kappa*}) = \mu^{s\kappa*}$. Now λ^{s1} cols $M^{s\kappa*}$ and hence $\lambda^{s1} = J^1 \lambda^{s1} J^{sK^{1s}} \lambda^{s1} = J^1 J^{sK^{1s}} \lambda^{s1} K^{1s} \lambda^{s1}$, which proves $55.5) \rightarrow 55.4)$. For $55.4) \rightarrow 55.1)$, we find from $\lambda^{s1} \overline{\text{rows}} M^{1\kappa}$ and the last condition in 55.4), that $J^1 \lambda^{s1} (J^{sK^{1s}} \lambda^{s1} - \epsilon_{\kappa}^1) = 0^{11}$. Now $K^{1s} M^{1\kappa s}$ by 21.5), and hence $(J^{sK^{1s}} \lambda^{s1} - \epsilon_{\kappa}^1)$ cols $M^{1\kappa}$. Since λ^{s1} is complete by rows $\mathcal{M}^{1\kappa}$, it follows at once that $J^{sK^{1s}} \lambda^{s1} = \epsilon_{\kappa}^1$.

For $O) \rightarrow 55.6)$, it suffices to prove the last condition in the latter. By 52.11), 32.27), and 32.23), we find $\lambda^{21} I^{2\lambda 1\lambda*}$ and $\epsilon_{\lambda*}^1 = \epsilon_{\kappa}^1$, $\epsilon_{\lambda}^2 = \epsilon_{\kappa*}^2$. From the last condition in D 51.1r, we find that K^{12} cols $\mathcal{M}^{2\lambda \lambda* 1\lambda*}$. Thus by 52.33) and 52.34), we have

$$J^2 K^{12} J^1 \lambda^{21} \mu^{1\kappa*} = J^1 (J^2 K^{12} \lambda^{21}) \mu^{1\kappa*} = J^1 \epsilon_{\kappa}^1 \mu^{1\kappa*} = \mu^{1\kappa*}.$$

This proves that $O)$ implies 55.6). For $55.6) \rightarrow 55.3)H$, it suffices to show that $J^1 \lambda^{21} K^{12} = \epsilon_{\kappa*}^2$. From $J^1 \lambda^{21} K^{12} H$, we find that $J^1 \lambda^{21} K^{12}$ is by cols $\mathcal{M}^{2\kappa*}$. Since K^{12} is by cols $M^{1\kappa*}$, we secure from the last condition in 55.6) that $J^2 K^{12} (J^1 \lambda^{21} K^{12}) = K^{12}$; hence

$$J^2 K^{12} (J^1 \lambda^{21} K^{12} - \epsilon_{\kappa*}^2) = O^{12}.$$

Since K^{12} comp $\overline{\text{rows}} \mathcal{M}^{2\kappa*}$, it follows that $J^1 \lambda^{21} K^{12} = \epsilon_{\kappa*}^2$. This completes the proof of T 55).

If we replace $(1, 2, K, \lambda)$ by $(2, 1, \lambda, K)$ in each of the statements 55.1) - 55.6), we secure, by virtue of 52.12), another set of necessary and sufficient conditions for a matrix λ^{21} to be a reciprocal of $K^{12} T^{12}$ in the sense defined in Chapter IV.

In view of 42.12) and 32.29), the theory as developed in the preceding chapter is related to the present one by an interchange of the role of K^{12} and λ^{21} . Thus the properties of the spaces associated with K^{12} and its reciprocal λ^{21} in this chapter can be deduced readily from § 15.

VITA

Yue Kei Wong was born in Chung-Shan District, Kwangtung, China. His 'primary school' education was completed in his native district. In 1918, he entered the middle school of Lingnan University (then Canton Christian College), and was graduated in 1922. The following Autumn, he came to the United States and studied in the University of Washington for one quarter. He entered the University of Chicago in October, 1924, and, except a period of illness, was in residence until 1930, studying under Professors Moore, Dickson, Bliss, Lane, Graves, Barnard, Slaught, Logsdon, MacMillan, and Bartky. He received his S. B. degree in 1927, and S. M. degree in 1929 from the University of Chicago.

The writer wishes to take this opportunity to express his gratitudes to Professor R. W. Barnard for suggestions in the preparation of this thesis, and to Professor E. H. Moore for the inspiration and encouragement that made possible the completion of this work.

